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ABSTRACT

Boundary condition perturbation theory has been limited to first-order analysis in transport theory. A full first-order boundary condition perturbation theory is developed for estimating the change in the system flux due to a perturbation in the boundary condition of the system. The derivation is developed in the context of energy dependent transport theory. Four example problems are presented in monoenergetic, one-dimensional slab geometry that verify the correctness of the expression.

1. INTRODUCTION

Boundary condition perturbation methods seek to find the change in the solution due to a perturbation in the boundary condition of the system. The parameters of interest are generally the eigenvalue, a functional of the solution or the solution itself. In diffusion theory, boundary condition methods have been limited to first-order (all references except McKinley) except that reported by McKinley and Rahnema (2000) in which high-order analysis is developed for eigenvalue problems. In transport theory, only the first-order change in the eigenvalue and the ratio of linear functionals of the solution have been formalized (Rahnema and Pomraning, 1981; Rahnema, 1996 and 1997; Rahnema and Ravetto, 1998).

In this paper a complete first-order theory will be developed for the homogeneous transport equation by deriving a formalism for the first-order change in the flux in response to a perturbation in the boundary condition of the system. Numerical examples will be given to verify the formalism.

2. DERIVATION OF FORMALISM

Before we derive the formalism for the first-order flux, we need to lay the groundwork by rederiving the first-order eigenvalue as previously shown in Rahnema (1996). Since the equations may become quite complicated, the following notation is used to simplify phase space integration:

$$< ab > \equiv \int_{\Re} d^3x \int_E dE \int_{4p} d^2\Omega ab \text{ and} \quad (1)$$

$$< ab >_s \equiv \int_{d\Re} d^2x \int_E dE \int_{4p} d^2\Omega ab \quad (2)$$

Where x is the spatial variable and E and W represent the neutron energy and direction. Symbols a and b are arbitrary operators and/or variables. The surface integral in Eq. (2) may further be modified by specifying certain angular directions $< ab >_{s,\hat{\Omega}\cdot\hat{n}<0}$, which limits the integral to only incoming directions.

The analysis starts off with the steady-state diffusion eigenvalue equation in a given domain $\Re$ for the unperturbed case:

$$H\vec{J}_0 = l_0 F\vec{J}_0 \quad (3)$$

where H and F are the transport operators defined as

$$H = \hat{\Omega}\cdot\nabla + s(\vec{x},E) - \int dE' \int_{4p} d^2\Omega' s_s(\vec{x};E',\hat{\Omega}' \to E,\hat{\Omega}) \text{ and} \quad (4)$$

$$F = \int dE' \int_{4p} d^2\Omega' n(\vec{x},E') s_f(\vec{x};E',\hat{\Omega}' \to E,\hat{\Omega}) \quad (5)$$

In the above equation s is the total cross section, s_s is the scattering cross section, n is the average number of neutrons released per fission, and s_f is the fission cross section.

The unperturbed normalized flux, $\vec{J}_0(\vec{x},E,\hat{\Omega})$, is assumed to be known and normalized such that $<\vec{J}_0>$ = 1. The known unperturbed eigenvalue, l_0, represents the inverse of the effective multiplication factor, k_{eff}. The flux is assumed to satisfy the following external boundary condition:

$$\vec{J}_0(\vec{x},E,\hat{\Omega}) = g_0\vec{J}_0(\vec{x},E,\hat{\Omega}_r), \vec{x} \in \P\Re,\ \hat{\Omega}\cdot\hat{n} < 0, \quad (6)$$

where $\hat{n}$ represents the outward unit normal at the boundary, g_0 is the fraction of neutrons that are reflected at the external boundary ($\partial\Re$ in the specular direction $\hat{\Omega}_r$ such that $\hat{\Omega}\cdot\hat{n} = -\hat{\Omega}_r\cdot\hat{n}$ and $(\hat{\Omega}\times\hat{\Omega}_r)\cdot\hat{n} = 0$).

When a small perturbation in the boundary condition is introduced, the transport equation is written as:

$$H\vec{J} = lF\vec{J}, \quad (7)$$

where $\vec{J}$ and λ are the perturbed normalized flux and eigenvalue, respectively.

Assuming that the flux and eigenvalue are smooth functions of the smallness parameter, e, then they may be expanded as

$$l = l_0 + e\,l_1 + O(e^2) \text{ and} \tag{8}$$

$$\bar{J} = \bar{J}_0 + e\bar{J}_1 + O(e^2) \tag{9}$$

where $<\bar{J}> = 1$. Integrating the above equation over all phase space and equating terms of e leads to $<\bar{J}_1> = 0$.

Using the expansion in Eqs. (8) and (9) in Eq. (7) yields:

$$H\bar{J}_0 + H\bar{J}_1 e = F(l_0\bar{J}_0 + l_1\bar{J}_0 e + l_0\bar{J}_1 e) + O(e^2)\,. \tag{10}$$

Combining terms of constant powers of e yields Eq. (3) and

$$(H - F\,l_0)\bar{J}_1 = F\,l_1\bar{J}_0\,. \tag{11}$$

Equation (11) also holds for the unnormalized flux as well.

Assume the perturbation takes the form of

$$\bar{J}_0(x, E, \hat{\Omega}) = (g_0 + e g_1)\bar{J}_0(\vec{x}, E, \hat{\Omega}_r), \vec{x} \in \P\Re, \hat{\Omega}\cdot\hat{n} < 0\,. \tag{12}$$

where g_1 is the perturbation in the fraction of reflected neutrons. This equation may be expanded with Eqs. (8) and (9) to give

$$\bar{J}_0 + e\bar{J}_1 = (g_0 + g_1)(\bar{J}_0 + e\bar{J}_1) + O(e^2) = 0, \vec{x} \in \P\Re, \hat{\Omega}\cdot\hat{n} < 0\,. \tag{13}$$

Combining terms of e yields Eq. (6) and

$$\bar{J}_1(\vec{x}, E, \hat{\Omega}) = g_0\bar{J}_1(\vec{x}, E, \hat{\Omega}_r) + g_1\bar{J}_0(\vec{x}, E, \hat{\Omega}_r), \vec{x} \in \P\Re, \hat{\Omega}\cdot\hat{n} < 0\,. \tag{14}$$

The first step is to solve for the first-order eigenvalue, l_1 by operating on Eq. (11) with $<\bar{J}_0^*, \cdot>$ where $\bar{J}_0^*$ is yet to be determined. After using the divergence theorem, change of variables and a vector identity manipulation on the resulting equation we get the following:

$$<\bar{J}_1\bar{J}_0^*\hat{\Omega}\cdot\hat{n}>_s + <\bar{J}_1(H^* - l_0F^*)\bar{J}_0^*> = l_1 <\bar{J}_0^* F\bar{J}_0> \tag{15}$$

where H* and F* are the adjoint operators:

$$H^* \equiv -\hat{\Omega}\cdot\nabla + s(\vec{x},E) - \int_{E'} dE' \int_{4p} d^2\Omega' s_s(\vec{x};E,\hat{\Omega}\to E',\hat{\Omega}') \text{ and} \tag{16}$$

$$F^* \equiv u(\vec{x},E)\int_{E'} dE' \int_{4p} d^2\Omega' s_f(\vec{x};E,\hat{\Omega}\to E',\hat{\Omega}'). \tag{17}$$

We define $\vec{J}_0^*$ as

$$(H^* - l_o F^*)\vec{J}_o^* = 0, \vec{x}\in\Re \text{ and} \tag{18}$$

$$\vec{J}_o^*(\vec{x},E,\hat{\Omega}) = g_o\vec{J}_o^*(\vec{x},E,\hat{\Omega}_r), \vec{x}\in\P\Re, \hat{\Omega}\cdot\hat{n}<0. \tag{19}$$

Omitting details (Rahnema, 1996), we use this definition of the adjoint to solve for l_1 in Eq. (15) to get:

$$l_1 = \frac{<g_1\vec{J}_0^*(\hat{\Omega})\vec{J}_0(\hat{\Omega}_r)\hat{\Omega}\cdot\hat{n}>_{s,\hat{\Omega}\cdot\hat{n}<0}}{<\vec{J}_0^* F\vec{J}_0>} \tag{20}$$

which is equivalent to Eq. (19) in (Rahnema, 1996).

Functionals of the flux solution is a previously explored topic for perturbed boundary condition (Rahnema, 1997). The flux can be defined as a functional of itself as

$$\begin{aligned}\vec{J}(\vec{r}_0) &= \frac{<d(\vec{x}-\vec{x}_0)d(E-E_0)d(\hat{\Omega}-\hat{\Omega}_0), j(\vec{x},E,\hat{\Omega})>}{<j(\vec{x},E,\hat{\Omega})>} \\ &\equiv \frac{<d(\vec{r}-\vec{r}_0), j(\vec{r})>_{\vec{r}}}{<j(\vec{r})>}\end{aligned} \tag{21}$$

where $\vec{r}$ represents all of phase space, $\vec{J}$ is the flux defined in Eq. (9), and d is the Dirac delta function. Variables with subscripts, 0, represent arbitrary phase space variables. The subscript, $\vec{r}$, in Eq. (21) indicates that integration is done over $\vec{r}$ and not $\vec{r}_0$.

Using the expansion of $\vec{J}$ in Eq. (21) gives

$$\begin{aligned}&(\vec{J}_0 + e\vec{J}_1)(<j_0> + e<j_1>) \\ &= <d(\vec{r}-\vec{r}_0), j_0(\vec{r})> + e<d(\vec{r}-\vec{r}_0), j_1(\vec{r})> + O(e^2).\end{aligned} \tag{22}$$

Multiplying out and equating terms of e yields the following equation for $\vec{J}_1$:

$$\vec{J}_1(\vec{r}_0) = \frac{< d(\vec{r} - \vec{r}_0), j_1(\vec{r}) > - \vec{J}_0 < j_1(\vec{r}) >}{< j_0(\vec{r}) >} \tag{23}$$

It is more convenient to rewrite this as

$$\vec{J}_1(\vec{x}_0, E_0, \hat{\Omega}_0) = \frac{< g(\vec{r}, \vec{r}_0), j_1(\vec{x}, E, \hat{\Omega}) >}{< j_0(\vec{x}, E, \hat{\Omega}) >} \tag{24}$$

where

$$g(\vec{r}, \vec{r}_0) \equiv d(\vec{r} - \vec{r}_0) - \vec{J}_o(\vec{r}_0). \tag{25}$$

Following the procedure for the solution of l_1, we operate on the unnormalized form of Eq. (11) with $<\Psi_0^*, \circ>$ where Ψ_0^* is a yet to be determined function. Following the same manipulations as we used before to derive Eq. (15), one obtains

$$< j_1 \Psi_0^* \hat{\Omega} \cdot \hat{n} >_s + < j_1 (H^* - l_0 F^*) \Psi_0^* >= l_1 < \Psi_0^* F j_0 >. \tag{26}$$

We define Ψ_0^* as the solution to

$$(H^* - l_0 F^*) \Psi_0^* = g \text{ and} \tag{27}$$

$$\Psi_0^*(\vec{x}, E, \hat{\Omega}) = g_0 \Psi_0^*(\vec{x}, E, \hat{\Omega}_r), \vec{x} \in \P\Re, \hat{\Omega} \cdot \hat{n} < 0. \tag{28}$$

By using the boundary condition in Eq. (14), which also applies to the unnormalized flux, we reduce Eq. (26) to

$$< g_1 \Psi_0^*(\hat{\Omega}) j_0(\hat{\Omega}_r) \hat{\Omega} \cdot \hat{n} >_{s, \hat{\Omega} \cdot \hat{n} < 0} + < j_1 g >= l_1 < \Psi_0^* F j_0 >. \tag{29}$$

Note that due to the Fredholm's Alternative (Courant, 1953) the following must be true for its existence of the adjoint $\Psi_0^*(\vec{x}, \vec{x}_0, E, E_0, \hat{\Omega}, \hat{\Omega}_0)$:

$$< j_0 g >= 0. \tag{30}$$

This can be verified by direct substitution of Eq. (25) into Eq. (30). Since the solution to Eq. (27) is not unique, we choose the following uniqueness condition for Ψ_0^*:

$$< \Psi_0^* F j_0 >= 0 =< \Psi_0^* F \vec{J}_0 >, \tag{31}$$

which when combined with Eqs. (24) and (29) yields

$$\vec{J}_1(\vec{x}_0, E_0, \hat{\Omega}_0) = \frac{-<g_1 \Psi_0^*(\hat{\Omega}) j_0(\hat{\Omega}_r)\hat{\Omega}\cdot\hat{n}>_{s,\hat{\Omega}\cdot\hat{n}<0}}{<j_0(\vec{x})>}. \quad (32)$$

One can use normalized fluxes in Eq. (32) since the normalized and unnormalized unperturbed fluxes are different by a scalar factor. Taking advantage of that and $<\vec{J}_0> = 1$, Eq. (32) is simplified as

$$\vec{J}_1(\vec{x}_0, E_0, \hat{\Omega}_0) = -<g_1 \Psi_0^*(\hat{\Omega})\vec{J}_0(\hat{\Omega}_r)\hat{\Omega}\cdot\hat{n}>_{s,\hat{\Omega}\cdot\hat{n}<0}. \quad (33)$$

This is the desired formalism for the first-order change in the solution.

3. EXAMPLE PROBLEMS

Four examples are provided to verify the formalism given by Eq. (33). For simplicity, the examples are all modeled as mono-energetic, one-dimensional slabs using discrete-ordinates (S-N) approximations. Scattering and fission reactions are assumed to be isotropic. The first three examples all use the same slab model with varying boundary conditions. The last example will show a two-region problem.

Example 1 - Vacuum Boundary Condition Slab

The first example is a simple slab of width 1 with $s = n s_f = 1$ and $s_s = 0.5$. The surfaces are exposed to a vacuum boundary condition ($g_0 = 0$ in Eq. (6)). We perturb the right side by changing eg_1 in Eq. (12) from 0.0 to 1.0. In addition, each case was run with a different order of the S-N approximation: S2, S4, and S16. The eigenvalue for each problem is compared in Table 1 below. The unperturbed (0^{th}-order approximation) eigenvalue for each case is simply the exact value for $eg_1 = 0.0$. Table 2 shows the errors as defined in Eq. (36) below for the 0^{th} and 1^{st}-order perturbation estimations. The x variable in Eq. (36) is an arbitrary data point. The 2^{nd}-order contribution in this equation can be estimated by a simple rewrite of Eq. (8) expanded to higher order and divided by g_1. Since g_1 and l_1 are constant as e varies, this term, shown in Eq. (37), should remain roughly constant when e is not too large. Finally, Table 3 displays the flux errors using the %RMS formula defined in Eq. (38). The equations used in the tables below are

$$\%error = 100\left(\frac{x_{exact} - x_{predicted}}{x_{exact}}\right), \quad (36)$$

$$\frac{l_2}{g_1^2} = \frac{l_{exact} - l_0 - e l_1}{e^2 g_1^2} + O(e), \text{ and} \quad (37)$$

$$\%RMS = \sqrt{\frac{\sum_{i=1}^{N}\left(\frac{x_{exact} - x_i}{x_{exact}}\right)^2}{N}}. \qquad (38)$$

Table 1 Eigenvalue ($l = 1/k_{eff}$) as a Function of Boundary Albedo and S-N Order

	Exact			First-Order		
eg_1	S2	S4	S16	S2	S4	S16
0.00	1.3867	1.1857	1.1174	1.3867	1.1857	1.1174
0.01	1.3799	1.1812	1.1136	1.3799	1.1812	1.1136
0.05	1.3534	1.1632	1.0986	1.3530	1.1630	1.0984
0.10	1.3208	1.1409	1.0799	1.3193	1.1403	1.0795
0.50	1.0872	0.9721	0.9368	1.0497	0.9588	0.9282
0.75	0.9639	0.8767	0.8542	0.8813	0.8453	0.8336
1.00	0.8557	0.7896	0.7775	0.7128	0.7318	0.7390

Table 2 Eigenvalue Error with Predicted 2nd-Order Term for Example 1

	0th-Order Error*			1st-Order Error*			2nd-Order Contribution**		
eg_1	S2	S4	S16	S2	S4	S16	S2	S4	S16
0.00	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	---	---	---
0.01	-0.5%	-0.4%	-0.3%	0.0%	0.0%	0.0%	0.206	0.201	0.180
0.05	-2.5%	-1.9%	-1.7%	0.0%	0.0%	0.0%	0.160	0.072	0.055
0.10	-5.0%	-3.9%	-3.5%	0.1%	0.1%	0.0%	0.155	0.058	0.040
0.50	-27.5%	-22.0%	-19.3%	3.4%	1.4%	0.9%	0.150	0.053	0.034
0.75	-43.9%	-35.2%	-30.8%	8.6%	3.6%	2.4%	0.147	0.056	0.037
1.00	-62.0%	-50.2%	-43.7%	16.7%	7.3%	4.9%	0.143	0.058	0.038

* See Eq. (36).

** See Eq. (37).

Table 3 Comparison of Flux Errors for Example 1

	0th-Order %RMS			1st-Order %RMS		
eg_1	S2	S4	S16	S2	S4	S16
0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.01	5.00	5.12	5.34	0.01	0.01	0.01
0.05	11.41	11.60	11.85	0.16	0.11	0.10
0.10	15.59	15.79	16.02	0.45	0.30	0.27
0.50	33.18	32.51	31.92	6.23	4.24	3.67
0.75	42.65	41.09	39.60	13.10	9.03	7.67
1.00	52.54	50.10	47.49	22.79	15.94	13.36

The first-order columns in Table 2 show that even when the boundary condition is perturbed to a reflective boundary condition ($eg_1 = 1.0$) that the errors may be as low as

4.9%. The nearly constant 2nd-order contribution in the last columns support that all of the 1st-order terms are accounted for by the formalism. The 2nd-order values for eg_1 = 0.01 are off due to the limited precision of the numerical methods. This example agrees very well with results presented in Rahnema (1996). The flux errors presented in Table 3 show the same general trend as the eigenvalue errors in Table 2. The only difference is the flux errors in general are about twice as large. This is expected since the eigenvalue is a global value and the fluxes are compared point by point.

Example 2 - Reflected Boundary Condition Slab

This example has the same setup as Example 1 but the unperturbed boundary condition is full specular reflection (g_0 = 1.0). The right boundary condition is again perturbed as before. The discrete ordinates used is S4. Table 4 contains the results for this case that were expressed in Tables 1 to 3 for the previous example.

The results for this case are very similar to the previous one. When eg_1 = -1.0, we have the vacuum boundary on one side with a spectral reflection on the other. This case is the same as Example 1 with eg_1 = 1.0. While the exact values are the same, the errors are much larger for Example 2. This shows that it is a larger perturbation change to go from reflective boundary to vacuum than the reverse. When eg_1 = 1.0 in this example, it may be interpreted as placing a highly fissionable material next to the slab. So for every neutron that escapes out the right side, two come back. The first-order errors are relatively small for such a large change.

Table 4 Eigenvalue and Flux Comparison with Specular Boundary Conditions

eg_1	λ_{exact}	l_0+el_1	0th-Order Error	1st-Order Error	0th-Order %RMS	1st-Order %RMS	2nd-Order Contr.
-1.00	0.7896	0.7606	36.7%	3.7%	1413.41	182.07	0.029
-0.75	0.7130	0.6955	29.9%	2.5%	65.89	8.34	0.031
-0.50	0.6385	0.6303	21.7%	1.3%	27.51	2.57	0.033
-0.10	0.5264	0.5261	5.0%	0.1%	3.75	0.08	0.034
-0.05	0.5131	0.5130	2.6%	0.0%	1.81	0.02	0.034
-0.01	0.5026	0.5026	0.5%	0.0%	0.35	0.00	0.034
0.00	0.5000	0.5000	0.0%	0.0%	0.00	0.00	---
0.01	0.4974	0.4974	-0.5%	0.0%	0.35	0.00	0.035
0.05	0.4871	0.4870	-2.7%	0.0%	1.71	0.02	0.035
0.10	0.4743	0.4739	-5.4%	0.1%	3.33	0.07	0.035
0.50	0.3784	0.3697	-32.1%	2.3%	14.18	1.65	0.035
0.75	0.3240	0.3045	-54.3%	6.0%	19.93	3.59	0.035
1.00	0.2736	0.2394	-82.7%	12.5%	25.35	6.24	0.034

Example 3 - Slab with Both Surfaces Perturbed

In an effort to tie in the previous two examples, Example 3 have the same setup but will have both boundary conditions perturbed by the same amount. Table 5 has the results of a vacuum boundary condition case perturbed to have some reflection. Table 6 shows the results of having full specular reflection and perturbing until both boundaries are modeled as a vacuum. Each perturbation step of Table 5 has a corresponding step in Table 6.

Table 5 Initial Vacuum Boundary Conditions Perturbed on Both Sides

eg_1	λ_{exact}	l_0+el_1	0^{th}-Order Error	1^{st}-Order Error	0^{th}-Order %RMS	1^{st}-Order %RMS	2^{nd}-Order Contr.
0.00	1.1857	1.1857	0.00%	0.00%	0.00	0.00	---
0.10	1.0980	1.0949	-7.99%	0.28%	20.80	0.71	0.304
0.25	0.9760	0.9588	-21.49%	1.76%	29.09	2.67	0.275
0.50	0.7953	0.7318	-49.08%	7.98%	37.05	7.15	0.254
0.75	0.6383	0.5049	-85.77%	20.89%	43.45	13.05	0.237
0.90	0.5533	0.3688	-114.28%	33.36%	47.25	17.19	0.228
1.00	0.5000	0.2780	-137.14%	44.40%	49.89	20.52	0.222

It is interesting to note that perturbing from the vacuum case, that the eigenvalue has the larger error than the flux while in the specular reflection case, the situation is reversed. This can be better understood by noting that the 0^{th}-order behaves in the same manner. The 2^{nd}-order terms also look fairly constant which supports that the formalism accounts for all first-order terms.

Table 6 Initial Specular Reflection Boundary Conditions Perturbed on Both Sides

eg_1	λ_{exact}	l_0+el_1	0^{th}-Order Error	1^{st}-Order Error	0^{th}-Order %RMS	1^{st}-Order %RMS	2^{nd}-Order Contr.
0.00	0.5000	0.5000	0.00%	0.00%	0.00	0.00	---
-0.10	0.5533	0.5521	9.64%	0.22%	4.23	0.12	0.121
-0.25	0.6383	0.6303	21.66%	1.25%	11.60	0.82	0.127
-0.50	0.7953	0.7606	37.13%	4.36%	29.38	3.93	0.139
-0.75	0.9760	0.8910	48.77%	8.71%	68.70	12.88	0.151
-0.90	1.0980	0.9691	54.46%	11.73%	147.45	31.40	0.159
-1.00	1.1857	1.0213	57.83%	13.87%	1195.91	207.99	0.164

Example 4 - Two-Region Slab Problem

This case takes the fuel slab from the previous examples with a thickness of 0.5 and adds another slab to the right side that is a reflector. The reflector is a pure scattering slab of thickness 0.5 and $s_t = 1$. The left and right sides are subjected to a vacuum boundary condition. In this example, the right side (reflector) boundary condition will be perturbed.

Table 7 Two-Region Slab with Reflector Boundary Condition Perturbed

εg_1	λ_{exact}	$l_0+\varepsilon l_1$	0^{th}-Order Error	1^{st}-Order Error	0^{th}-Order %RMS	1^{st}-Order %RMS	2^{nd}-Order Contr.
0.00	0.9185	0.9185	0.0%	0.0%	0.00	0.00	---
0.01	0.9151	0.9151	-0.4%	0.0%	7.38	0.03	0.237
0.05	0.9015	0.9016	-1.9%	0.0%	15.52	0.08	-0.006
0.10	0.8842	0.8846	-3.9%	0.0%	20.55	0.14	-0.037
0.15	0.8666	0.8677	-6.0%	-0.1%	23.96	0.21	-0.048
0.30	0.8114	0.8168	-13.2%	-0.7%	30.67	0.52	-0.060
0.50	0.7322	0.7491	-25.4%	-2.3%	36.74	1.03	-0.068
0.75	0.6228	0.6644	-47.5%	-6.7%	43.45	1.78	-0.074
1.00	0.5000	0.5796	-83.7%	-15.9%	51.21	2.66	-0.080
1.25	0.3612	0.4949	-154.3%	-37.0%	61.87	3.71	-0.086
1.50	0.2034	0.4102	-351.6%	-101.7%	78.11	5.06	-0.092
1.75	0.0225	0.3255	-3981.0%	-1346.3%	105.30	7.04	-0.099

The perturbation continues up to $\varepsilon g_1 = 1.75$. Perturbing the boundary condition past this point may lead to a diverging solution. The errors of the first-order perturbations are very good even for such a large change away from the initial case. The 2^{nd}-order term does not appear to be constant in this example. Below $\varepsilon g_1 = 0.3$, numerical error due to the limited precision of the answer accounts for the error in computing the 2^{nd}-order term. Past $\varepsilon g_1 = 0.5$, the higher-order terms become more important so the computed term can no longer be assumed to be constant.

4. CONCLUSIONS

In this paper, an expression was developed for a first-order flux solution to due to a perturbation in the boundary condition of the system. The formalism is derived in energy dependent transport theory. Numerical examples are provided in mono-energetic one-dimensional geometry for four example problems.

This work may be extended to higher-order theory. Such a theory could greatly increase the accuracy of the system solution and have application to coarse-mesh transport method by allowing accurate homogenization heterogeneous node (coarse-mesh).

Rahnema and Ravetto (1998) showed that, to first-order, a perturbation in the boundary of the system could be converted into an equivalent perturbation in the boundary condition of the system. Any methods that extend the degree of precision of boundary condition perturbation also may be used to improve boundary perturbation problems.

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