STATISTICAL ANALYSIS OF OVERDISPERSION IN NUCLEAR COUNTING DISTRIBUTIONS

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ABSTRACT

Experimental evidence suggests that nuclear counting data are frequently overdispersed with respect to the Poisson distribution. A stochastic model is described, based on sequential excess-fluctuating processes, leading to overdispersion. A distribution function for overdispersed Poisson is derived, which belongs to a family of generalized hypergeometric factorial moment distributions by Kemp and Kemp. The formula for the dispersion coefficient is also derived. A limiting case of only one excess-fluctuating process is considered. In that limit, the overdispersed Poisson distribution transforms into three known overdispersed distributions: beta-Poisson, negative binomial, and overdispersed Gaussian, depending on the range of parameters. Three cases of nuclear counting distributions data are fitted using the limiting distributions, which performed well within their ranges of validity. The sources of the overdispersion were inferred individually and attributed to the excess fluctuations of the detection systems or sequential radioactive decay. The overdispersion has implications for detection limits and confidence intervals in nuclear counting.

1. INTRODUCTION

According to an idealized model, the Poisson distribution describes fluctuations in nuclear statistics (Bateman, 1910). Nearly exact Poisson statistics have been reached under stringent conditions (Cannizzaro *et al.*, 1978; Concas and Lissia, 1997; Silverman *et al.*, 2000). It is known from experimental data, however, that the Poisson model is often inadequate. A dispersion coefficient of counts $\delta_x = \mu_2/\mu$, where μ is the mean and μ_2 is the variance, is frequently used as a test for Poisson statistics (where it is equal to 1). One is often faced with a so called overdispersion in nuclear counting distributions caused by either excess fluctuations (Semkow, 1999a, 1999b, 2001) or sequential decay (Inkret *et al.*, 1990), both leading to $\delta_x > 1$. The main effect of the overdispersed statistics is that it decreases the precision of nuclear measurements (Currie, 1972). In addition, the detection limits increase (Tries, 1997). Therefore, these considerations are extremely important for nuclear counting applications, as they allow for a proper evaluation of confidence intervals as well as detection limits of the measurements in the presence of overdispersion.

This work represents a distributional approach to the overdispersed nuclear statistics. The elements of overdispersed distributions have been introduced to nuclear counting before (Pazdur, 1976; Müller, 1978; Fröhner, 1997). A stochastic model of sequential processes has been proposed to describe overdispersion in nuclear statistics (Semkow, 1999a, 1999b). The processes typically encountered leading to radioassay are production, extraction, survival, decay, and detection of radioactive atoms. The total probability of a count is thus a product of the probabilities of individual processes. The individual probabilities may exhibit excess fluctuations in the course of the experiments. The statistics become overdispersed if the average probabilities change from measurement to measurement. In this case the fluctuations are called the random Lexis fluctuations (Johnson *et al.*, 1993). If the fluctuations do not change the average probabilities (so called random Poisson fluctuations), the statistics remain Poisson (Breitenberger, 1955).

In this work, the stochastic model is generalized by including also excess fluctuations of quantities other than the probabilities such as the number of radioactive atoms. This leads to the random Ottestad (1943) fluctuations. In Section 2.1, a distribution function is derived by combining the Lexis and Ottestad fluctuations. This function is called the overdispersed Poisson (OP) distribution. It is shown that the OP distribution belongs to a family of distributions by Kemp and Kemp (1974). In addition, a formula for the dispersion coefficient is derived. In Section 2.2, a limited version of the general model is considered, in which only a single excess-fluctuating process is assumed in a given experiment. The OP distribution then leads to the three known statistical distributions which can describe overdispersed nuclear statistics: beta-Poisson (BP), negative binomial (NB), and overdispersed Gaussian (OG). The NB distribution has been used in nuclear statistics before (Pazdur, 1976; Müller, 1978; Fröhner, 1997), while the OG distribution is implied in many of the χ^2 methods (Currie, 1972; Tries, 1997, 2000; Tries *et al.*, 1999; Almeida *et al.*, 2000). The BP distribution has been introduced to nuclear statistics by Semkow (2001).

To verify the proposed models and describe procedures for handling overdispersed nuclear data, 13 cases of nuclear counting distributions were analyzed (Semkow, 2001): 12 of them measured and 1 from the literature. The data consisted of the measurements of either the backgrounds or radioactive standards on a variety of instruments for environmental radioactivity and bioassay. The measurements included Lucas cell, gas proportional detectors, $\beta\gamma$ coincidence, NaI detector, liquid scintillation counter, as well as scaler data by Müller (1978). Out of the 13 cases studied, 11 were found overdispersed and only 2 consistent with the Poisson statistics. In this work, 3 representative overdispersed cases are discussed for the BP, NB, and OG distributions in Section 3. The measurements and the fitting procedure to the experimental data, based on the higher-moment analysis, are described. The sources of the overdispersion are individually inferred from the fits and a treatment of outliers is discussed. Section 4 provides the summary and conclusions followed by a nomenclature to facilitate the reading.

2. THEORY OF OVERDISPERSED POISSON DISTRIBUTION

2.1 Multiple Excess-Fluctuating Processes

Let us consider a stable or radioactive atom, which may be subjected to a process consisting of sequential and conditional events such as transmutation, extraction, survival, decay, detection, or any combination thereof. Let us assign event probabilities p_i, i=1,…,r, where r is the number of p processes. The probability of the combined process to occur is $P = \Pi^{r}_{i=1} p_i$. The mean number of atoms undergoing the combined process is given by

$$\mu = NP, \tag{1}$$

where N is the number of original atoms. By the requirement of a Poisson process, $N >> 1$ and $P << 1$, which is satisfied by at least one of $p_i << 1$.

A specific example of such a sequential process is a decay of a long lived radionuclide, starting with N radioactive atoms. To register a count, two events have to occur: radioactive decay, described by the probability of decay $p_1 = \lambda t << 1$, where λ is a decay constant and t is the measurement time, as well as a probability of detection $0 < p_2 = \varepsilon < 1$, where ε is the detector efficiency. The mean number of counts is given by $\mu = Np_1p_2$, while the individual counts x fluctuate according to a Poisson distribution (Bateman, 1910). In experiments, however, N, p_1, and p_2 do not have to be constant and can exhibit excess fluctuations. For instance, N could be fluctuating due to different samples or fluctuating background; p_1 could fluctuate due to time fluctuations of an electronic timer; p_2 could fluctuate due to sample positioning, temperature, pressure, or electronics setup. If p_1 or p_2 exhibit excess fluctuations according to the Lexis scheme or N according to the Ottestad scheme, the statistics of counts become overdispersed.

The mean number of counts $\mu = N\lambda t\varepsilon$ can be factored out in different ways, such as $\mu = At\varepsilon$, where $A = N\lambda$ is the disintegration rate, or $\mu = \rho t$, where $\rho = N\lambda\varepsilon$ is the counting rate. Then, the excess fluctuations of either A, ρ, t, or ε can be considered depending on the experimental situation.

This picture can be generalized by factoring into a constant M all processes which do not exhibit excess fluctuations. The excess-fluctuating processes are factored into two products of variables, so that Eq. (1) is rewritten as

$$\mu = M \prod_{i=1}^{r} p_i \prod_{i=r+1}^{s} q_i \,, \tag{2}$$

where r is the number of the p-type (probability type) processes, s is the total number of processes, and $s - r$ is the number of the q-type processes. It has been shown that the processes of the p-type can be described by a beta random variable (Semkow, 1999b).

The processes of the q-type are described by a (standardized) gamma random variable (Dennis and Patil, 1984), and any scaling factors of the standardization enter the M constant. For instance, if $q=ct$ for time fluctuations, where c is a scaling factor, then $M = \rho/c$. The mean values and the variation coefficients v for the p and q processes are given by, respectively,

$$\bar{p} = \frac{a}{b}, \qquad v_p^2 = \left(\frac{\sigma_p}{\bar{p}}\right)^2 = \frac{b-a}{a(b+1)}, \tag{3a}$$

$$\bar{q} = a\,, \qquad v_q^2 = \left(\frac{\sigma_q}{\bar{q}}\right)^2 = \frac{1}{a}, \tag{3b}$$

where a and b are parameters and σ is the standard deviation.

In the following, the probability distribution function is derived for the overdispersed Poisson distribution. One starts with the probability generating function for the Poisson distribution

$$G(z) = {}_0F_0\left[;;M\prod_{i=1}^{r} p_i \prod_{i=r+1}^{s} q_i\,(z-1)\right], \tag{4}$$

where ${}_kF_l$ is the generalized hypergeometric function defined as follows (Slater, 1966)

$${}_kF_l[(a);(b);z] = \sum_{j\geq 0} \frac{\prod_{i=1}^{k}(a_i)_j}{\prod_{i=1}^{l}(b_i)_j} \frac{z^j}{j!} \;. \tag{5}$$

In Eq. (5), $(a)_j = \Pi^j_{i=1}(a+i-1)$ abbreviates an ascending factorial, $(a)_0 = 1$, k is the number of a parameters, l is the number of b parameters, and z is the argument.

Equation (4) is integrated r times over the beta random variable and $s - r$ times over the gamma random variable. The following recursive integrals for the generalized hypergeometric function are used (Slater, 1966; Johnson *et al.*, 1993)

$${}_{k+1}F_{l+1}[(a);(b);z] = \int_0^1 {}_kF_l[(a);(b);pz]\frac{p^{a_{k+1}-1}(1-p)^{b_{l+1}-a_{k+1}-1}}{B(a_{k+1},b_{l+1}-a_{k+1})}dp\,, \tag{6a}$$

$${}_{k+1}F_l[(a);(b);z] = \int_0^\infty {}_kF_l[(a);(b);qz]\frac{q^{a_{k+1}-1}e^{-q}}{\Gamma(a_{k+1})}dq\,, \tag{6b}$$

where B is the beta function and Γ is the gamma function. The integrations result in the generating function for the overdispersed Poisson process

$$G(z) = {}_sF_r\left[(a);(b);M(z-1)\right]. \tag{7}$$

The distributions having the generating function given by the generalized hypergeometric function with an argument proportional to 1 – *z* were discovered by Kemp and Kemp (1974) and are named the generalized hypergeometric factorial moment distributions (GHFD) (Johnson *et al.*, 1993). The distribution function of counts *x* can be derived from Eq. (7) by expanding *G*(*z*) in powers of *z* and collecting the terms for z^x yielding

$$OP(x\,|\,M,(a),(b)) = \frac{\mu'_{[x]}}{x!}\,{}_sF_r\left[(a+x);(b+x);-M\right], \tag{8}$$

where $\mu'_{[x]}$ is a factorial moment of the *x*th order defined below. This is the overdispersed Poisson distribution and the main result of this work.

The dispersion is studied by performing a moment analysis. By substituting *z* = 1 + *y* in Eq. (7), one obtains a factorial moment generating function *G*(1+*y*). Then, by expanding it in a Taylor series, the factorial moments $\mu'_{[j]}$ are coefficients of $y^j/j!$

$$\mu'_{[j]} = \frac{\prod_{i=1}^{s}(a_i)_j}{\prod_{i=1}^{r}(b_i)_j}M^j. \tag{9}$$

Having the factorial moments, one can calculate the mean μ, and variance μ_2 from (Johnson *et al.*, 1993)

$$\mu = \mu'_{[1]}, \qquad \mu_2 = \mu'_{[2]} + \mu - \mu^2, \tag{10}$$

as well as a dispersion coefficient of counts δ_x. The results are

$$\mu = M\frac{\prod_{i=1}^{s}a_i}{\prod_{i=1}^{r}b_i}, \tag{11a}$$

$$\delta_x = \frac{\mu_2}{\mu} = 1 + \mu\left[\prod_{i=1}^{r}\left(1+v_{p_i}^2\right)\prod_{i=r+1}^{s}\left(1+v_{q_i}^2\right) - 1\right]. \tag{11b}$$

It is seen that the mean (Eq. (11a)) is just an average of Eq. (2), using Eqs. (3). It is also seen from Eq. (11b) that $\delta_x > 1$. The term after the 1st plus sign in Eq. (11b) is the overdispersion term. Since it is proportional to μ, the larger the mean, the larger the overdispersion. The overdispersion term depends on the variation coefficients v_i of the excess-fluctuating processes given by Eqs. (3) and their correlations. As expected, the overdispersion disappears when there are no excess fluctuations, $v_i = 0$.

2.2 Single Excess-Fluctuating Process

The generalized model of overdispersion presented in Section 2.1 could be verified in experiments having multiple stages, such as solar neutrino or neutron activation analysis. For simple counting, it can be assumed that only a single extra-fluctuating process is present.

As a first case, it is considered that the counting efficiency ε has excess fluctuations, which is a process of the *p*-type ($0 < p = \varepsilon < 1$). From Eq. (11a), the mean counts is $\mu = Ma/b$, where *M* is the mean disintegrations in a counting experiment. Under such circumstances, the OP distribution (Eq. (8)) transforms to a beta-Poisson distribution of Gurland (1958), given below, where the 1st Kumer transformation (Johnson *et al.*, 1993) was also applied

$$BP(x \mid M, a, b) = \frac{M^x}{x!} e^{-M} \frac{(a)_x}{(b)_x} {}_1F_1[b-a; b+x; M]. \tag{12}$$

Let us consider the excess-fluctuating process of the *q*-type. One has from Eq. (11a) $\mu = Ma$, where *a* is the mean of the standardized gamma fluctuating variable (compare Eq. (3b)). Then the OP distribution (Eq. (8)) transforms to a negative binomial distribution

$$NB(x \mid \mu, a) = \frac{(\mu / a)^x (a)_x}{x!(1 + \mu / a)^{a+x}}. \tag{13}$$

Note that the BP distribution is more general than the NB distribution.

An interesting situation happens when $p << 1$ for the excess-fluctuating process. It has been shown that, under such conditions, the BP distribution transforms to a NB distribution (Semkow, 2001). Therefore, the time fluctuations can be treated equivalently by either the NB distribution with $q = ct$, or the BP distribution with $p = \lambda t << 1$.

For a large number of counts one obtains the Gaussian limit. Specifically the conditions are $\mu >> 1$ and $(x - \mu) / \mu << 1$. In the Gaussian limit, both NB and BP lead to an overdispersed Gaussian distribution (Semkow, 2001)

$$OG(x \mid \mu, a) = \frac{1}{\sqrt{2\pi\mu(1 + \mu / a)}} e^{-\frac{(x-\mu)^2}{2\mu(1+\mu/a)}}. \tag{14}$$

The relationships between the limiting overdispersed distributions are summarized in Fig. 1. Also indicated on the left side of Fig. 1 is the transition between the classical Poisson and Gaussian distributions, recognized early (Bortkiewicz, 1913).

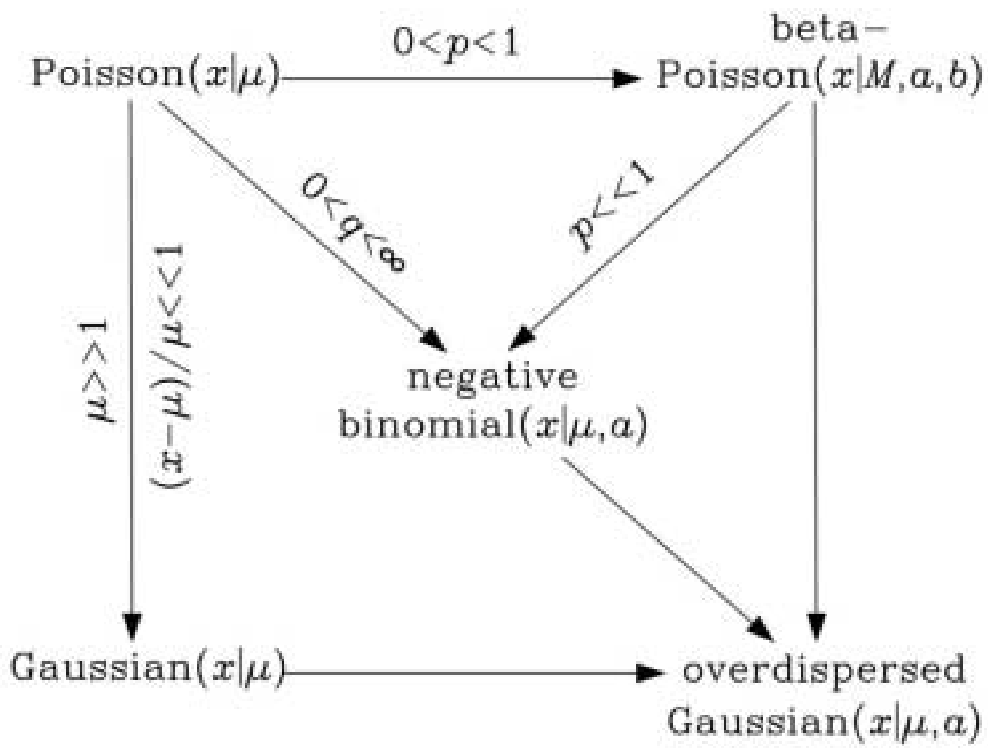

Fig. 1. Relationships between the classical and overdispersed distributions in nuclear statistics. *p*, *p* process; *q*, *q* process; *x*, counts; μ, mean counts; *M*, *a*, *b*, distribution parameters.

All of the limiting overdispersed distributions have a dispersion coefficient of the form (Eq. (11b))

$$\delta_x = 1 + \mu v^2 , \tag{15}$$

where v is the variation coefficient of the excess-fluctuating process: v_p in the case of BP, v_q in case of NB, and any of them in case of OG.

3. FITS TO NUCLEAR COUNTING DISTRIBUTIONS

In this Section, three representative cases of the overdispersed nuclear counting data are fitted using the 3 limiting overdispersed distributions described in Section 2.2: BP, OG, and NB. The data and the fits are summarized in Table 1. The total time in Table 1 is the time span of the whole experiment, while μ_2, μ_3, and UT-prob defined below.

Table 1. Statistical data evaluations and distribution fits.

Item	Case 1	Case 2	Case 3
Distribution	Beta-Poisson	Ovdsp. Gaussian	Neg. binomial
Figure	2	3	4
Detection	Gas prop.	Gas prop.	Scaler[a]
Radionuclide	α bkg	^{239}Pu	n/a
Number obs.	2579	2504	385360
Count time	50 m	10 m	0.1 s
Total time	4.4 y	4.4 y	11 h
δ_x	1.21	1.30	1.04
μ	1.72	3.61×10^{3}	41.0
μ_2	2.07	4.69×10^{3}	42.7
μ_3	2.92	----	----
v^2	0.119	8.55×10^{-5}	9.91×10^{-4}
a	8.20	1.20×10^{4}	1.01×10^{3}
b	455	----	----
UT-prob	1.94×10^{-12}	8.51×10^{-23}	1.23×10^{-69}

a) Data by Müller (1978).

Cases 1 and 2 concern the measurements of α background as well as ^{239}Pu α standard on a gas proportional counter. They are long-term measurements, *i.e.*, were made typically once or more a day for several years. Cases 1 and 2 were not measured with the goal of studying the overdispersion. Rather, they represent routine performance checks of the counter in a service laboratory. Case 3 consists of short (0.1 s), consecutive measurements with a scaler, by Müller (1978). These data are, therefore, short-term.

In this work, a higher-moment analysis is used. Suppose $i = 1,\ldots,n$ measurements of counts x_i were made ($n >> 1$). The statistical moments are calculated directly from the data as follows. Mean counts

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i \ . \qquad (16a)$$

Central moments of counts (μ_2 is the variance)

$$\mu_j = \frac{1}{n}\sum_{i=1}^{n} \left(x_i - \mu\right)^j \ , \qquad j \geq 2 \ . \qquad (16b)$$

Such defined moments represent sample (or experimental) moments. The moments (or the parameters derived from them) entering the distribution functions (Eqs. (12-14)) are population moments. The distinction between the two is another subject in statistics and is not considered here for the following reasons. The overdispersion is the 1st order effect studied here. Since $n >> 1$ (Table 1), the sample moments should approximate the population moments very well and any deviation would be a 2nd order effect compare to the overdispersion effect.

The dispersion coefficient δ_x and the experimental moments are given in Table 1. It is seen, that all 3 cases are overdispersed ($\delta_x > 1$). To determine a significance of this statement, χ^2 tests were performed. While the χ^2 test applies strictly to normal data, it can be applied approximately to the Poisson data as well (Haight, 1967). $\chi_0^2 = n\delta_x$ values were calculated for each case studied followed by the upper-tail probability (UT-prob in Table 1) for the χ^2 distribution with $n - 1$ degrees of freedom. UT-prob is a probability that a random, nonoverdispersed sample would give a $\chi^2 \geq \chi_0^2$. UT-prob provides the significance of a χ^2 test: the lower UT-prob, the less significance there is to the null hypothesis that the data are not overdispersed. It is seen from Table 1 that there is virtually no chance that the data are not overdispersed.

The data for case 1 yielded small mean counts, so the BP distribution is appropriate. BP is a 3-parameter distribution: M, a, b (Eq. (12)), where $M = b\mu/a$ (Eq. (11a)). μ is obtained directly from the data (Eq. (16a)), while a and b are calculated from the equations given below (Semkow, 2001), using experimental μ_2, μ_3 from Eq. (16b).

$$a = 2\mu \frac{\mu(\mu_3 - \mu_2) - (\mu_2 - \mu)(\mu_2 + \mu - \mu^2)}{\mu_2^2(4\mu - 3) - \mu_3(\mu^2 + \mu - \mu_2) - \mu(\mu - 1)(5\mu_2 - 2\mu)},$$

$$b = 2\frac{\mu(\mu_3 - \mu_2) - (\mu_2 - \mu)(\mu_2 + \mu - \mu^2)}{2\mu_2^2 - \mu(\mu_2 + \mu_3)}. \tag{17}$$

The counting data data are plotted as frequencies in Fig. 2 (points). Also plotted are the BP distribution according to Eq. (12) (solid curve), as well as a simple Poisson distribution for the same mean counts (dash-dot curve). It is seen that BP fits the data very well, while Poisson is not adequate. The semi-logarithmic plot in Fig. 2a emphasizes the tails of the distributions; the mode region is revealed better on the linear scale of Fig. 2b. One can also see two open points, based on one or two measurements, which were considered outliers. They do not conform to the statistics and are therefore systematic problems. Including the outliers would increase δ_x from 1.21 to 1.27 and would bias the statistics.

The data for the case 2 satisfy the conditions for the OG distribution, which is a 2-parameter distribution: μ and a. The mean is calculated from the experimental data (Eq. (16a)) and a from the second moment (Eq. (16b)) using the equation below

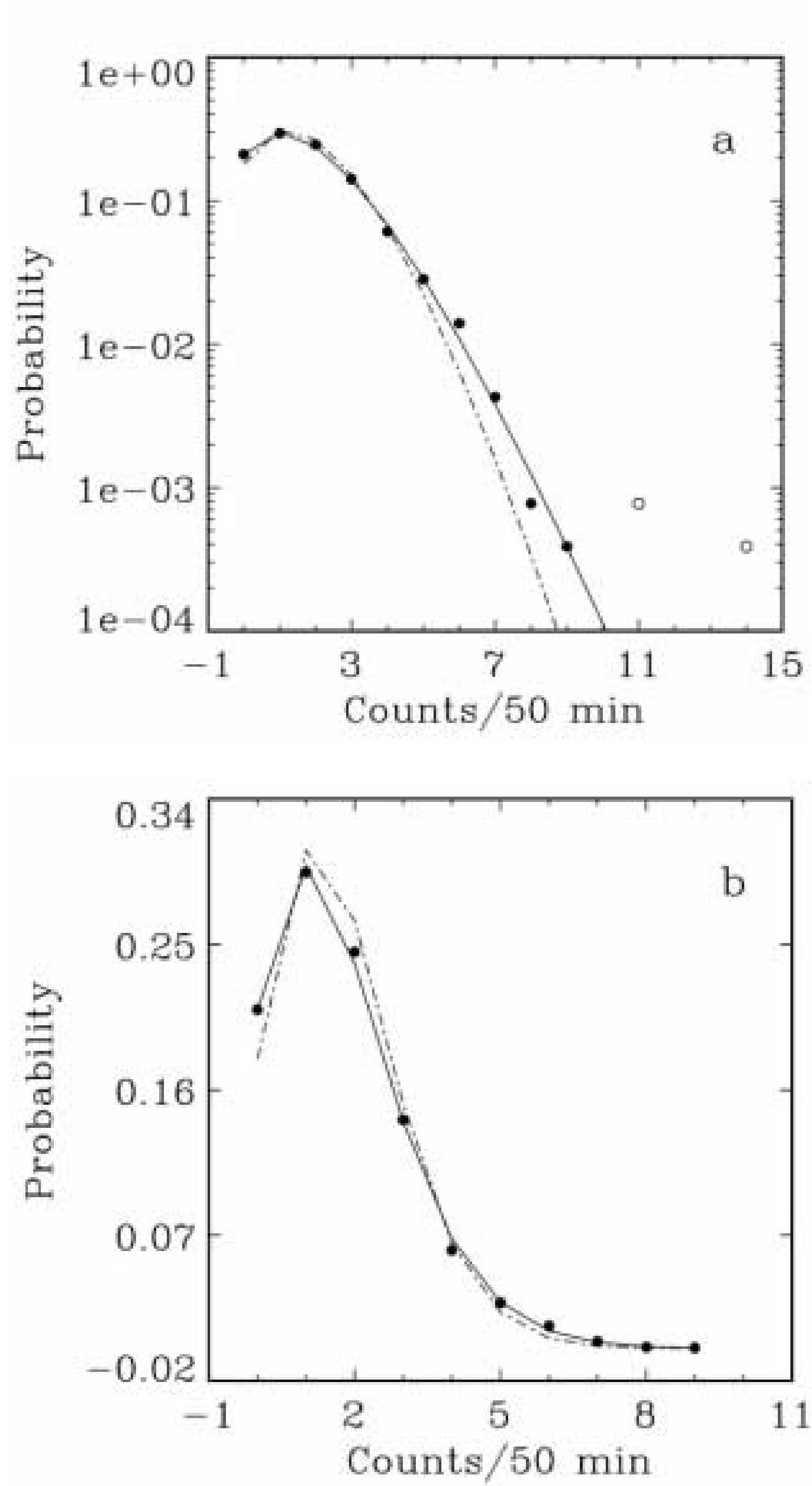

Fig. 2. Probability distribution functions for α background in a gas proportional counter. Solid points, data; open points, outliers; solid curve, beta-Poisson; dash-dot curve, Poisson. a) semilog scale; b) linear scale.

$$a = \frac{\mu}{\mu_2 - \mu}. \tag{18}$$

The counting data data are plotted as frequencies in Fig. 3 (points). Also plotted are the OG distribution according to Eq. (14), as well as a nonoverdispersed Gaussian distribution for the same mean counts. It is seen that OG fits the data better than the Gaussian.

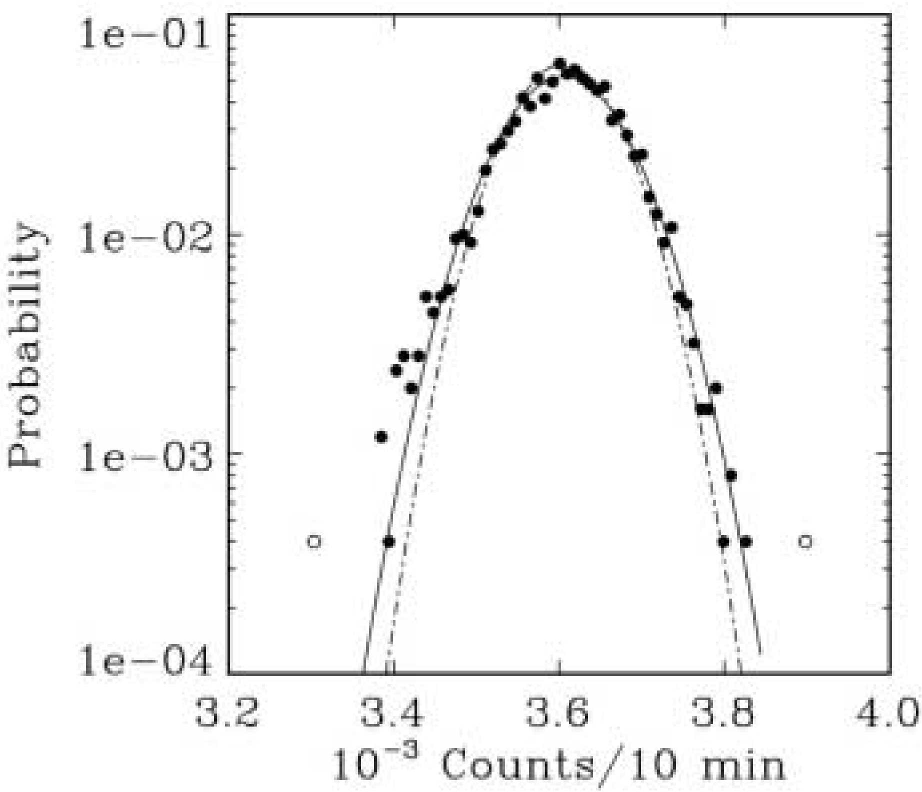

Fig. 3. Probability distribution functions for ^{239}Pu α standard in a gas proportional counter. Solid points, data; open points, outliers; solid curve, overdispersed Gaussian; dash-dot curve, Gaussian.

Also given in Table 1 is the square of the variation coefficient for the excess-fluctuating process v^2, which is calculated directly from the data using Eq. (15). The v^2 may tell something about the excess-fluctuating source. For both cases 1 (α background) and 2 (^{239}Pu α standard) the same counter was used. A common source with similar v would imply a much higher dispersion coefficient for the standard than for the background, by Eq. (15). It is seen from Table 1 that δ_x are similar, while the *v* differ by a factor of 37. The small *v* deduced from the standard measurements may indicate instrumental effects such as long-term high voltage instabilities of the counter. The much higher *v* of the background suggests a different source. Since the counter is equipped with an active cosmic shield, the background is caused predominately by radon and daughters in the air as well as intrinsic impurities of the detector materials from U and Th decay chains. The α background fluctuations could be due to a fluctuating concentration of radon. This is probably not the case, however, since the air in the laboratory is well ventilated regardless of the day, time, or season. Therefore, the overdispersion in the α background may be due to sequential decay of Rn or U and Th series, especially, since each measurement time was 50 min. If this is the case, then the BP distribution can apparently describe the statistics of sequential decay (Inkret *et al.* 1990) as well.

For scaler data by Müller (1978) (case 3), fluctuations of a timer are responsible for overdispersion because of a short measurement time. This data set is of an exceptional statistical quality due to the large number of observations (~ 3.8 x 10^5). Therefore, in spite of a small $\delta_x = 1.04$, the data are considerably overdispersed according to the χ^2 test (see Table 1). The time fluctuations can be treated as a *q*-variable or as a *p*-variable. Since $p = \lambda t << 1$, the BP distribution transforms to the NB distribution anyway. NB is a 2-parameter distribution: μ and *a*, so calculation of the parameters is similar to the OG described above. Figure 4a shows the mode region of the data (points), the NB fit (solid curve), and the Poisson fit for the same mean counts (dashed-dot curve). Clearly, NB represents a good fit, while Poisson misses the data. Figure 4b depicts a left wing region of the distribution. Careful examination confirms the adequacy of the NB fit.

4. SUMMARY AND CONCLUSIONS

There is an increasing evidence from the literature suggesting the presence of overdispersion in nuclear counting data with respect to the Poisson distribution. The overdispersed data are characterized with the dispersion coefficient $\delta_x > 1$.

A stochastic model is proposed to describe the overdispersed Poisson process. The overdispersion is caused by a sequence of the excess-fluctuating processes of the two types. The *p* process is described by a beta random variable and it derives from the Lexis fluctuations. The *q* process is described by a standardized gamma random variable and it derives from the Ottestad fluctuations. The distribution function for the overdispersed Poisson is derived (Eq. (8)) as well a formula for the dispersion coefficient (Eq. (11b)). This distribution belongs to a GHFD family by Kemp and Kemp (1974).

The general model could be applied to experiments consisting of several processes such as production, extraction, survival, decay, and detection of radioactive atoms. Some possible experiments could involve neutron activation analysis, or radiochemical solar neutrino. For radioactivity counting, when often only one excess-fluctuating process can be assumed in a given experiment, the OP distribution transforms into three limiting overdispersed distributions: BP, NB and OG. Three sets of experimental data were fitted with the limiting distributions using moment analysis. All

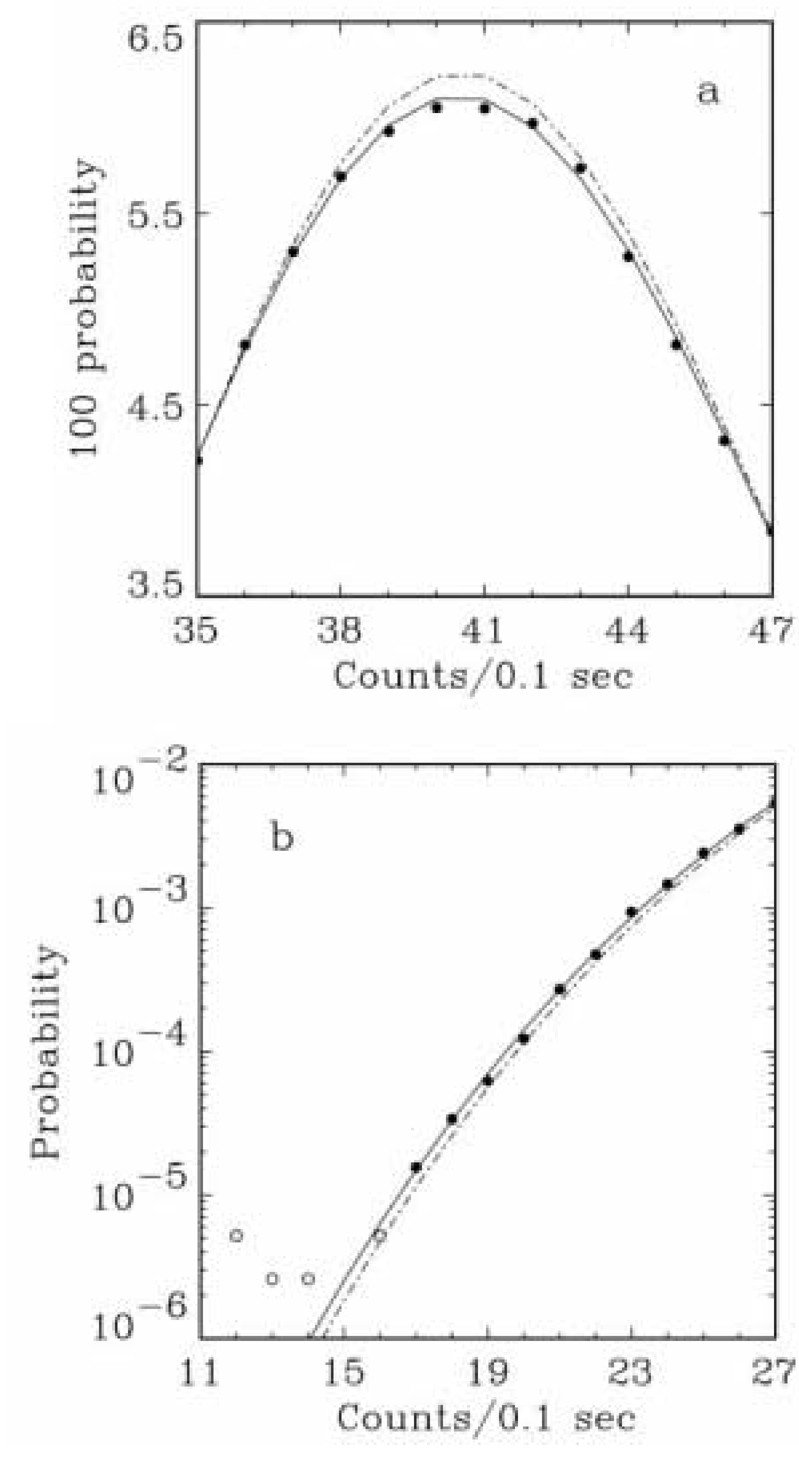

Fig. 4. Probability distribution functions for scaler data by Müller (1978). Solid points, data; open points, outliers; solid curve, negative binomial; dash-dot curve, Poisson. a) distribution mode; b) distribution left wing.

the limiting distributions performed well within their ranges of validity. An unexpected result was found that the BP distribution can also describe the statistics of sequential radioactive decay inferred for one set of data.

Two distributions: NB and OG can be easily fitted to limited data, since they require only the first two experimental moments and the formulas are simple. The BP distribution has a more complicated expression to calculate which, in itself, is not a limitation with modern computers. However, it also needs a third moment from the data, which requires many more measurements to determine with a sufficient accuracy. This limitation could be removed by using a maximum likelihood fit. As shown in this work, the distributional approach can help in rejecting the systematic outliers from the data.

Overdispersion is often neglected in nuclear statistics. However, recognition and treatment of the overdispersion are extremely important for proper evaluation of the confidence intervals and detection limits. Consider, for instance, the fact that the detectors are standardized infrequently in a typical service laboratory (*e.g.*, yearly) while the performance is verified with a check source at prescribed intervals (*e.g.*, daily) (ANSI, 1996). The overall uncertainty of the standardization should include any overdispersion of the performance, and the detection limits should include overdispersion in long-term background measurements, if present.

NOMENCLATURE

a, a_i	distribution parameter, of the *i*th process
$(a)_j$	*j*th ascending factorial
A	disintegration rate
b, b_i	distribution parameter, of the *i*th process
B	beta function
BP	beta-Poisson
c	scaling factor
${}_kF_l$	generalized hypergeometric function
G	generating function
GHFD	generalized hypergeometric factorial moment distribution
n	number of measurements
N	number of atoms
NB	negative binomial
M	distribution parameter
OG	overdispersed Gaussian
OP	overdispersed Poisson
p, p_i, $\bar{p}$	event probability, of the *i*th process, mean value
P	probability of the combined process
q, q_i, $\bar{q}$	standardized gamma variable, of the *i*th process, mean value
r	number of p processes
s	total number of processes
t	measurement time
UT-prob	upper-tail probability

v, v_i, v_p, v_q	variation coefficient, of the *i*th, *p*, *q* process
x, x_i	number of counts, in the *i*th measurement
y, *z*	variables
Γ	gamma function
δ_x	dispersion coefficient of counts
ε	detector efficiency
λ	radioactive decay constant
μ	mean counts
μ_2	variance of counts
μ_3	third moment of counts
$\mu'_{[j]}$	factorial moment of the *j*th order
ρ	counting rate
σ_p, σ_q	standard deviation of the *p*, *q* process
χ^2, χ_0^2	chi-square variable, experimental

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