STUDIES ON NODAL METHODS FOR THE TIME-DEPENDENT CONVECTION-DIFFUSION EQUATION

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ABSTRACT

Two methods for solving the time-dependent, two-dimensional, convection-diffusion equation were studied and their spatial and temporal orders of convergence were compared. The first is a full space-time nodal integral method (NIM) in which both the temporal and spatial operators are discretized using the nodal integral approach. The second method is a hybrid finite-difference/NIM (FD/NIM) in which the temporal operator is discretized using a backward finite-difference in time, and then the spatial operator, along with the terms resulting from discretizing the temporal operator, is discretized using the nodal integral approach. The asymptotic spatial and temporal truncation errors of the two methods were studied using two test problems with exact analytical solutions. The results show that the full space-time NIM is second order in both space and time, while the FD/NIM is second order in space but only first order in time, and that for fixed accuracy requirements the former is more computationally efficient than the latter. Both methods are stable and exhibit no cell Peclet number restrictions.

1. INTRODUCTION

The time-dependent convection-diffusion equation of heat transfer in two space dimensions is

$$\alpha\frac{\partial T(x,y,t)}{\partial t}+u(x,y,t)\frac{\partial T(x,y,t)}{\partial x}+v(x,y,t)\frac{\partial T(x,y,t)}{\partial y}$$
$$-\frac{\partial^2 T(x,y,t)}{\partial x^2}-\frac{\partial^2 T(x,y,t)}{\partial y^2}-s(x,y,t)=0 \quad (1)$$

where $\alpha=\rho C_P/\mathrm{K}$, $\mathrm{u(x,y,t)} = \rho C_P u'(x,y,t)/\mathrm{K}$, $\mathrm{v(x,y,t)} = \rho C_P v'(x,y,t)/\mathrm{K}$, $\mathrm{s(x,y,t)} = S(x,y,t)/\mathrm{K}$, ρ = the density, C_p = the specific heat, K = the thermal conductivity, $\mathrm{u(x,y,t)}$ = the x-velocity component, $\mathrm{v(x,y,t)}$ = the y-velocity component, and $S(x,y,t)$ = the heat source. This equation describes many important phenomena, in addition to heat transfer, in engineering, geology, meteorology and physics. Moreover, it is widely used as a mathematical model, especially in the evaluation of numerical methods, to represent the salient features of linear and nonlinear fluid flow equations, including the Navier-Stokes equations (Patankar, 1980). For these reasons, the evaluation of a potential method for the numerical solution of the fluid flow equations often begins with the study of the method in the context of the convection-diffusion equation.

Because of their simplicity, finite-difference methods [see for example: (Roache, 1976) and (Fletcher, 1988)] are widely used for the numerical solution of partial differential equations, including the equations of fluid flow. However, because of their high accuracy and high computational efficiency, nodal integral methods (NIMs) (Azmy, 1983) now have been developed for the efficient numerical solution of many steady-state and time-dependent partial differential equations, including those of fluid flow (Azmy, 1983), (Wilson, 1988), (Elnawawy, 1990), (Esser, 1993), (Decker, 1993), (Michael, 1993), (Rizwan-uddin, 1997) and (Michael, 2001a). Because of this activity, and the fact that a previously developed NIM for the two-dimensional steady-state convection-diffusion equation did not exhibit any cell Peclet number restrictions (Michael, 1993) and was shown to be considerably more accurate and computationally efficient (achieves specified accuracy in dramatically smaller amounts of CPU time) than an excellent state-of-the-art finite-difference method (Michael, 2001b), a study of the asymptotic spatial and temporal convergence of a full space-time NIM for the convection-diffusion equation in a two-dimensional space-time setting was warranted. Further, since an alternative to a full space-time NIM, in which the time variable is treated using the nodal integral approach, is a hybrid finite-difference/NIM (FD/NIM) in which the time derivative is backward differenced and the resulting 2-D spatial operator is treated using the nodal integral approach, a study of the spatial and temporal convergence of a hybrid FD/NIM also was warranted. In fact, these studies were carried out as a prelude to developing a primitive-variable NIM for the time-dependent Navier-Stokes equations (Michael, 2001a), to determine whether the development of that method for those nonlinear equations should be based on the ideas that underlie a full space-time NIM or those that underlie a hybrid FD/NIM.

The formulation of the full space-time NIM in the 2-D setting is discussed in Sec. 2, and that for the hybrid FD/NIM is discussed in Sec. 3, where the differences from a previously developed hybrid method (Elnawawy, 1990) also are pointed out. The numerical results that establish the orders of the spatial and temporal errors of the two methods are presented in Sec. 4. Finally, in Sec. 5 conclusions on the relative advantages of the two methods are drawn.

2. THE FULL SPACE-TIME NIM

The formulation of the full space-time NIM, which is an extension of some of the works cited above, begins from Eq. (1), the time-dependent, 2-D, convection-diffusion equation. Although the NIMs are discussed here in the context of the heat transfer equation in two spatial dimensions, they of course are directly applicable to general linear convection-diffusion problems, and the extension to 3-D is straightforward. The space-time computational domain is first divided into $I \times J \times N$ computational elements or "nodes," each of volume $[-a_{i,j},+a_{i,j}]\times[-b_{i,j},+b_{i,j}]\times[-\tau,+\tau]$ (where the subscript *n* on τ is suppressed). Then, transverse averaging Eq. (1) over y and t locally, within a space-time node, yields

$$\frac{d^2\overline{T}^{y,t}(x)}{d\,x^2}-\overline{u}^{y,t}_{i,j}(x)\frac{d\overline{T}^{y,t}(x)}{d\,x}=\overline{S}^{y,t}_{i,j}(x) \tag{2}$$

where

$$\overline{u}^{y,t}_{i,j}(x)\equiv\frac{1}{2\tau}\int_{-\tau}^{+\tau}dt\,\frac{1}{2b_{i,j}}\int_{-b_{i,j}}^{+b_{i,j}}dy\;u(x,y,t)$$

and the "pseudo-source"

$$\overline{S}^{y,t}_{i,j}(x)\equiv\frac{1}{2\tau}\int_{-\tau}^{+\tau}dt\,\frac{1}{2b_{i,j}}\int_{-b_{i,j}}^{+b_{i,j}}dy\left[\alpha\frac{\partial T(x,y,t)}{\partial t}+v(x,y,t)\frac{\partial T(x,y,t)}{\partial y}-\frac{\partial^2 T(x,y,t)}{\partial y^2}-s(x,y,t)\right]$$

Here the integral of the product of $u(x,y,t)$ and $d\,T(x,y,t)/dx$ has been approximated by expanding both factors in Legendre polynomials in y and t and truncating at leading order, which leads to the average of the product being replaced by the product of the average. This results in a second-order error at this point (Michael, 2001b). Next, $\overline{u}^{y,t}_{i,j}(x)$ and $\overline{S}^{y,t}_{i,j}(x)$ are expanded in Legendre polynomials and truncated at leading order to obtain

$$\frac{d^2\overline{T}^{y,t}(x)}{d\,x^2}-u_{i,j}\,\frac{d\overline{T}^{y,t}(x)}{d\,x}=\overline{S}^{y,t}_{oi,j} \tag{3}$$

where

$$u_{i,j}\equiv\overline{u}^{y,t,x}_{i,j}\;\equiv\frac{1}{2a_{i,j}}\int_{-a_{i,j}}^{+a_{i,j}}dx\,\frac{1}{2\tau}\int_{-\tau}^{+\tau}dt\,\frac{1}{2b_{i,j}}\int_{-b_{i,j}}^{+b_{i,j}}dy\;u(x,y,t)$$

and

$$\overline{S}^{y,t}_{oi,j}\equiv\frac{1}{2a_{i,j}}\int_{-a_{i,j}}^{+a_{i,j}}dx\,\frac{1}{2\tau}\int_{-\tau}^{+\tau}dt\,\frac{1}{2b_{i,j}}\int_{-b_{i,j}}^{+b_{i,j}}dy\left[\alpha\frac{\partial T(x,y,t)}{\partial t}+v(x,y,t)\frac{\partial T(x,y,t)}{\partial y}-\frac{\partial^2 T(x,y,t)}{\partial y^2}-s(x,y,t)\right]$$

These approximations also lead to an overall second order error (Michael, 2001b). Analogously, transverse averaging Eq. (1) over x and t yields

$$\frac{d^2\overline{T}^{x,t}(y)}{d\,y^2} - \overline{v}_{i,j}^{x,t}(y)\frac{d\overline{T}^{x,t}(y)}{d\,y} = \overline{S}_{i,j}^{x,t}(y) \tag{4}$$

where

$$\overline{v}_{i,j}^{x,t}(y) \equiv \frac{1}{2\tau}\int_{-\tau}^{+\tau} dt\,\frac{1}{2a_{i,j}}\int_{-a_{i,j}}^{+a_{i,j}} dx\ v(x,y,t)$$

and

$$\overline{S}_{i,j}^{x,t}(y) \equiv \frac{1}{2\tau}\int_{-\tau}^{+\tau} dt\,\frac{1}{2a_{i,j}}\int_{-a_{i,j}}^{+a_{i,j}} dx\left[\alpha\frac{\partial T(x,y,t)}{\partial t} + u(x,y,t)\frac{\partial T(x,y,t)}{\partial x} - \frac{\partial^2 T(x,y,t)}{\partial x^2} - s(x,y,t)\right]$$

and expanding $\overline{v}_{i,j}^{x,t}(y)$ and $\overline{S}_{i,j}^{x,t}(y)$ in Legendre polynomials and truncating at leading order gives

$$\frac{d^2\overline{T}^{x,t}(y)}{d\,y^2} - v_{i,j}\frac{d\overline{T}^{x,t}(y)}{d\,y} = \overline{S}_{oi,j}^{x,t} \tag{5}$$

where

$$v_{i,j} \equiv \overline{v}_{i,j}^{x,t,y} \equiv \frac{1}{2b_{i,j}}\int_{-b_{i,j}}^{+b_{i,j}} dy\,\frac{1}{2\tau}\int_{-\tau}^{+\tau} dt\,\frac{1}{2a_{i,j}}\int_{-a_{i,j}}^{+a_{i,j}} dx\ v(x,y,t)$$

and

$$\overline{S}_{oi,j}^{x,t} \equiv \frac{1}{2b_{i,j}}\int_{-b_{i,j}}^{+b_{i,j}} dy\,\frac{1}{2\tau}\int_{-\tau}^{+\tau} dt\,\frac{1}{2a_{i,j}}\int_{-a_{i,j}}^{+a_{i,j}} dx\left[\alpha\,\frac{\partial T(x,y,t)}{\partial t} + u(x,y,t)\frac{\partial T(x,y,t)}{\partial x} - \frac{\partial^2 T(x,y,t)}{\partial x^2} - s(x,y,t)\right]$$

Finally, transverse averaging Eq. (1) over y and x yields

$$\frac{d\overline{T}^{y,x}(t)}{dt} = \overline{S}_{i,j}^{y,x}(t) \tag{6}$$

where

$$\overline{S}^{y,x}_{i,j}(t) \equiv -\frac{1}{\alpha_{i,j}}\frac{1}{2a_{i,j}}\int_{-a_{i,j}}^{+a_{i,j}} dx \frac{1}{2b_{i,j}}\int_{-b_{i,j}}^{+b_{i,j}} dy \left[u(x,y,t)\frac{\partial T(x,y,t)}{\partial x} + v(x,y,t)\frac{\partial T(x,y,t)}{\partial y} - \frac{\partial^2 T(x,y,t)}{\partial x^2} - \frac{\partial^2 T(x,y,t)}{\partial y^2} - s(x,y,t) \right]$$

Expanding $\overline{S}^{y,x}_{i,j}(t)$ in Legendre polynomials in t and truncating at leading order (which results in a second order error in time) gives

$$\frac{d\overline{T}^{y,x}(t)}{dt} = \overline{S}^{y,x}_{oi,j} \qquad (7)$$

where

$$\overline{S}^{y,x}_{oi,j}(t) \equiv -\frac{1}{\alpha_{i,j}}\frac{1}{2\tau}\int_{-\tau}^{+\tau} dt \frac{1}{2a_{i,j}}\int_{-a_{i,j}}^{+a_{i,j}} dx \frac{1}{2b_{i,j}}\int_{-b_{i,j}}^{+b_{i,j}} dy \left[u(x,y,t)\frac{\partial T(x,y,t)}{\partial x} + v(x,y,t)\frac{\partial T(x,y,t)}{\partial y} - \frac{\partial^2 T(x,y,t)}{\partial x^2} - \frac{\partial^2 T(x,y,t)}{\partial y^2} - s(x,y,t) \right]$$

Expressions for the diffusion fluxes at node interfaces are obtained by analytically solving Eqs. (3) and (5) within a computational element or node. Then, the analytical solutions to Eq. (3) for the $\overline{T}^{y,t}(+a_{i,j})$ and those to Eq. (5) for the $\overline{T}^{x,t}(+b_{i,j})$ are coupled in neighboring nodes using the continuity of the diffusion fluxes and of the temperatures at node interfaces to arrive at the final equations for the discrete-variable spatial unknowns $\overline{T}^{y,t}(+a_{i,j})$ and $\overline{T}^{x,t}(+b_{i,j})$ in terms of the pseudo-sources $\overline{S}^{y,t}_{oi,j}$ and $\overline{S}^{x,t}_{oi,j}$. Finally, expressions for $\overline{S}^{y,t}_{oi,j}, \overline{S}^{x,t}_{oi,j}$ and $\overline{T}^{y,x}_{i,j}(+\tau)$ in terms of the node surface-averaged temperatures and the true source $s(x,y,t)$, are derived using three equations per node. The first equation is obtained by integrating the convection-diffusion equation over x, y and t within a node (which ensures conservation). The other two are obtained by imposing the uniqueness of the node-averaged temperature ($\overline{T}^{y,t,x}_{i,j} = \overline{T}^{x,t,y}_{i,j} = \overline{T}^{y,x,t}_{i,j}$), regardless of the order of averaging. That is, regardless of the original orders in which they were twice averaged, then obtained analytically from Eqs. (3), (5) and (7), respectively, and finally averaged over the third variable. The expressions thus obtained for the pseudo-sources $\overline{S}^{y,t}_{oi,j}$ and $\overline{S}^{x,t}_{oi,j}$ are substituted into the equations resulting from the coupling of neighboring nodes to obtain the final set of discretized equations for the full space-time NIM. After these equations for $\overline{T}^{y,t}(+a_{i,j})$ and $\overline{T}^{x,t}(+b_{i,j})$ are solved iteratively, the expression for $\overline{T}_{i,j}^{y,x}(+\tau)$ is used to advance the calculations in time from one time level ($-\tau$) to the next ($+\tau$).

3. THE HYBRID FD/NIM

In the hybrid FD/NIM, the time derivative is approximated by introducing a backward finite-difference scheme in Eq. (1), to arrive at

$$\frac{\alpha[T(x,y,t_n)-T(x,y,t_{n-1})]}{\Delta t}+u(x,y,t_n)\frac{\partial T(x,y,t_n)}{\partial x}+v(x,y,t_n)\frac{\partial T(x,y,t_n)}{\partial y}$$
$$-\frac{\partial^2 T(x,y,t_n)}{\partial x^2}-\frac{\partial^2 T(x,y,t_n)}{\partial y^2}-s(x,y,t_n)=0 \qquad (8)$$

where the notation $-\tau$ and $+\tau$ for the previous and current time levels used in Sec. 2 on the full space-time NIM has been replaced by t_{n-1} and t_n to conform with the standard finite-difference notation. Equation (8) can be rewritten in the following two forms

$$\frac{\partial^2 T(x,y,t_n)}{\partial x^2}-u(x,y,t_n)\frac{\partial T(x,y,t_n)}{\partial x}-\alpha' T(x,y,t_n)$$
$$=-\alpha' T(x,y,t_{n-1})+S_1(x,y,t_n) \qquad (9)$$

and

$$\frac{\partial^2 T(x,y,t_n)}{\partial y^2}-v(x,y,t_n)\frac{\partial T(x,y,t_n)}{\partial y}-\alpha' T(x,y,t_n)$$
$$=-\alpha' T(x,y,t_{n-1})+S_2(x,y,t_n) \qquad (10)$$

where

$$S_1(x,y,t_n)\equiv-\left[\frac{\partial^2 T(x,y,t_n)}{\partial y^2}-v(x,y,t_n)\frac{\partial T(x,y,t_n)}{\partial y}+s(x,y,t_n)\right]$$

and

$$S_2(x,y,t_n)\equiv-\left[\frac{\partial^2 T(x,y,t_n)}{\partial x^2}-u(x,y,t_n)\frac{\partial T(x,y,t_n)}{\partial x}+s(x,y,t_n)\right]$$

Transverse-averaging Eq. (9) in the y-direction locally, within a spatial computational element or spatial node, yields

$$\frac{d^2\overline{T}^y(x,t_n)}{dx^2}-\overline{u}^y_{i,j}(x,t_n)\frac{d\overline{T}^y(x,t_n)}{dx}-\alpha'_{i,j}\,\overline{T}^y(x,t_n)$$
$$=-\alpha'_{i,j}\,\overline{T}^y(x,t_{n-1})+\overline{S}^y_{1i,j}(x,t_n) \qquad (11)$$

where

$$\alpha'_{i,j}\equiv\frac{\alpha}{\Delta t}; \qquad \overline{u}^y_{i,j}(x,t_n)\equiv\frac{1}{2b_{i,j}}\int_{-b_{i,j}}^{+b_{i,j}}dy\,u(x,y,t_n)$$

and

$$\overline{S}^y_{1i,j}(x,t_n)\equiv\frac{-1}{2b_{i,j}}\int_{-b_{i,j}}^{+b_{i,j}}dy\left[\frac{\partial^2T(x,y,t_n)}{\partial y^2}-v(x,y,t_n)\frac{\partial T(x,y,t_n)}{\partial y}+s(x,y,t_n)\right],$$

and the factors in the product $u(x,y,t_n)\partial T(x,y,t_n)/\partial x$ have been expanded in Legendre polynomials in y and truncated at leading order, resulting in a second-order spatial error. At this point the development here differs from that of an earlier hybrid FD/NIM (Elnawawy, 1990) in which the term containing the unknown at the n-th time level $-\alpha'_{i,j}\,\overline{T}^y(x,t_n)$ that results from the backward differencing of the time derivative, was left on the right-hand side of the equation and included with the pseudo-source term. Here, it is included on the left-hand side so that it does not have to be approximated as a constant, and of course the solution to Eq. (11) for $\overline{T}^y(x,t_n)$ is quite different than it is when that term is left on the right-hand side. Equation (11) can be further simplified by expanding $\overline{u}^y(x,t_n)$ in Legendre polynomials in x and truncating at zeroth order, which maintains the spatial discretization error at second order (Michael, 2001b). The y-averaged pseudo-source term $\overline{S}^y_{1i,j}(x,t_n)$ in Eq. (11) also is expanded in Legendre polynomials and truncated at leading order; however, to ensure that the method will exactly solve a problem with a solution that is linear in x and constant in y and t, the linear term in the Legendre expansion of $\overline{T}^y(x,t_{n-1})$ is retained, leading to

$$\frac{d^2\overline{T}^y(x,t_n)}{dx^2}-u_{i,j}(t_n)\frac{d\overline{T}^y(x,t_n)}{dx}-\alpha'_{i,j}\,\overline{T}^y(x,t_n)$$
$$=-\alpha'_{i,j}\,\overline{T}^y_{oi,j}(t_{n-1})-\alpha'_{i,j}\,\overline{T}^y_{1i,j}(t_{n-1})\,(x/a_{i,j})+\overline{S}^y_{1oi,j}(t_n) \qquad (12)$$

where

$$u_{i,j}(t_n) \equiv \frac{1}{4a_{i,j}b_{i,j}} \int_{-a_{i,j}}^{+a_{i,j}} dx \int_{-b_{i,j}}^{+b_{i,j}} dy\, u(x,y,t_n)\,;$$

$$\overline{T}^{y}_{1i,j}(t_{n-1}) \equiv \frac{3}{2a_{i,j}} \int_{-a_{i,j}}^{+a_{i,j}} dx \frac{x}{a_{i,j}} \frac{1}{2b_{i,j}} \int_{-b_{i,j}}^{+b_{i,j}} dy\, T(x,y,t_{n-1})$$

and

$$\overline{S}^{y}_{1oi,j}(t_n) \equiv \frac{-1}{4a_{i,j}b_{i,j}} \int_{-a_{i,j}}^{+a_{i,j}} dx \int_{-b_{i,j}}^{+b_{i,j}} dy \left[\frac{\partial^2 T(x,y,t_n)}{\partial y^2} - v(x,y,t_n)\frac{\partial T(x,y,t_n)}{\partial y} + s(x,y,t_n) \right]$$

That the first order term in the Legendre expansion of $\overline{T}^{y}(x,t_{n-1})$ is needed can be explained by the following argument. Consider the special case in which $T(x,y,t) = x$ is the exact solution to Eq. (1). Then, of course, from Eq. (1) it follows that for this to be a solution $s(x,y,t)$ must equal $u(x,y,t)$. Now, for $\overline{T}^{y}(x,t_n) = x$, which follows from $T(x,y,t) = x$, to be an exact solution to Eq. (12) the linear term $-\alpha'_{i,j}\,\overline{T}^{y}_{1i,j}(t_{n-1})(x/a_{i,j})$ must be retained. Of course, $\overline{S}^{y}_{1oi,j}(t_n)$ must equal $-u_{i,j}(t_n)$ also, but this follows from $s(x,y,t) = u(x,y,t)$ (and the fact that $\overline{S}^{y}_{1oi,j}(t_n) = -\overline{s}^{y}_{o}(t_n)$ since $T(x,y,t)$ is constant in y).

Transverse averaging Eq. (10) in the x-direction and using steps analogous to those above yields

$$\frac{d^2\overline{T}^{x}(y,t_n)}{dy^2} - \overline{v}^{x}_{i,j}(t_n)\frac{d\overline{T}^{x}(y,t_n)}{dy} - \alpha'_{i,j}\,\overline{T}^{x}(y,t_n)$$
$$= -\alpha'_{i,j}\,\overline{T}^{x}_{oi,j}(t_{n-1}) - \alpha'_{i,j}\,\overline{T}^{x}_{1i,j}(t_{n-1})\,(y/b_{i,j}) + \overline{S}^{x}_{2oi,j}(t_n) \qquad (13)$$

where

$$\overline{S}^{x}_{2oi,j}(t_n) \equiv \frac{-1}{4a_{i,j}b_{i,j}} \int_{-a_{i,j}}^{+a_{i,j}} dx \int_{-b_{i,j}}^{+b_{i,j}} dy \left[\frac{\partial^2 T(x,y,t_n)}{\partial y^2} - v(x,y,t_n)\frac{\partial T(x,y,t_n)}{\partial y} + s(x,y,t_n) \right]$$

Equations (12) and (13) are solved analytically for $\overline{T}^{y}(x,t_n)$ and $\overline{T}^{x}(y,t_n)$, and the diffusion fluxes at node interfaces are derived from these solutions. Then, the continuity of these fluxes and of the temperatures at node interfaces is used to couple the solutions to Eq. (12) for neighboring nodes and to couple the solutions to Eq. (13) for neighboring nodes. Finally, expressions for the pseudo-sources $\overline{S}^{y}_{1oi,j}(t_n)$ and $\overline{S}^{x}_{2oi,j}(t_n)$ are obtained using two conditions per node. The first is that the node average of Eq. (8) over x and y be zero, which ensures conservation. The second is that the node-averaged temperature be unique ($\overline{T}^{y,x}_{i,j}(t_n) = \overline{T}^{x,y}_{i,j}(t_n)$), regardless of the order of averaging, solving the once-averaged equation, and then calculating the average in the second spatial variable. The expressions for the pseudo-sources are then substituted into the equations that result from the coupling of neighboring nodes to yield the final set of discretized hybrid FD/NIM equations for the node-surface temperatures $\overline{T}^{y}(+a_{i,j}, t_n)$ and $\overline{T}^{x}(+b_{i,j}, t_n)$, which are solved iteratively.

The values of $\overline{T}^{y}_{1i,j}(t_{n-1})$ and $\overline{T}^{x}_{1i,j}(t_{n-1})$ for the first time step follow from the initial conditions. For subsequent time steps they are obtained by evaluating expressions for $\overline{T}^{y}_{1i,j}(t_n)$ and $\overline{T}^{x}_{1i,j}(t_n)$ derived by taking the y- and x-moments, respectively, of the solutions to Eqs. (12) and (13), written in terms of the node-surface temperatures $\overline{T}^{y}(+a_{i,j}, t_n)$ and $\overline{T}^{x}(+b_{i,j}, t_n)$.

4. NUMERICAL CONVERGENCE STUDIES

The only approximations made in the formulation of the full space-time NIM (Sec. 2) were the local (within node) expansions in the space and time variables of the two factors in each of the convection terms, and of each of the pseudo-sources; and the truncation of all these expansions at leading order led to second order errors in both space and time. Hence, this method should exhibit second order convergence in the node size in both space and time. Conversely, because the hybrid FD/NIM is formulated by starting with a simple backward finite-difference approximation to the time derivative, it should have a first order error in the time step; but since the spatial discretization is developed using the nodal integral approach, the expansions in the space variables of the factors in the convection terms and of the pseudo-sources led to a second order error in space. Thus this method should be second order in space, but first order in time. To study and compare the convergence in space and time of the two methods, two simple test problems were developed. Since numerical errors in convection-diffusion problems often vary with the direction of the flow field and almost always vary with the Peclet number $Pe \equiv \rho C_P v / K$, the test problems were based on a recirculating flow velocity field, and they were solved for a wide range of Peclet numbers. The velocity field, known *a priori*, is derived from the stream function $\Psi = (1-x^2)(1-y^2)$ and is shown on the 2×1 computational domain in Fig. 1 (Smith, 1982), and the Peclet number was varied from $Pe = 1$ to $Pe = 10.^{20}$

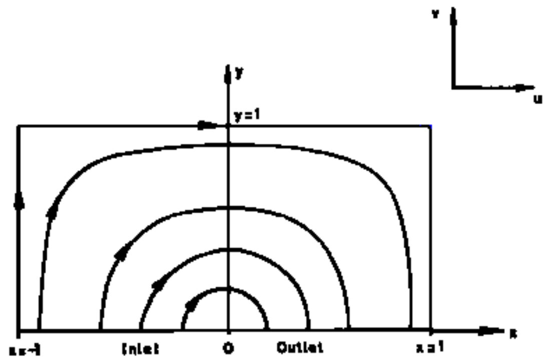

Fig. 1 The geometry and flow field for the computational domain of the two test problems.

Two simple test problems with exact analytical solutions were created to study the orders of the temporal and spatial errors of the two methods. To analyze the temporal error, a space-independent exact analytical solution $T(x, y, t) = T_o + T_o t^4$ was chosen, and to establish the order of the spatial error, a time-independent exact analytical solution $T(x, y, t) = T_o + T_o x^4 + T_o y^4$ was used. Expressions for the true source terms $s(x, y, t)$ were generated by substituting the exact analytical temperature distributions into the time-dependent, two-dimensional, convection-diffusion equation, Eq. (1). These exact solutions also were used to obtain the associated boundary and initial conditions.

Since the average of the temperature over the spatial surface at a time level, $\overline{T}_{i,j}^{y,x}(+\tau)$ for the full space-time NIM and $\overline{T}_{i,j}^{y,x}(t_n)$ for the hybrid FD/NIM, is a computed quantity in both methods, the error in it was used as a basis for comparison. Specifically, the error in this quantity, used in the comparisons, was the percent average L_1 relative error E_n defined as

$$E_n = \left\{ \sum_{j=1}^{J} \sum_{i=1}^{I} \left| \frac{\overline{T}_{C\,i,j}^{y,x}(t_n) - \overline{T}_{EX\,i,j}^{y,x}(t_n)}{\overline{T}_{EX\,i,j}^{y,x}(t_n)} \right| \right\} \times \frac{100}{I \times J}$$

where $\overline{T}_{C\,i,j}^{y,x}(t_n) \equiv$ the calculated value of the x- and y-averaged temperature over the (i,j)-th node at time level t_n $(+\tau$ in the full space-time NIM formulation in Sec. 2), and $\overline{T}_{EX\,i,j}^{y,x}(t_n) \equiv$ the exact value of the temperature averaged in x and y over the (i,j)-th node at time level t_n.

The numerical solutions were obtained at each time level using the full space-time NIM by starting from an initial guess (taken as the converged value at the previous time level), doing one Gauss-Seidel iteration on the equations for the $\overline{T}^{y,t}(a_{i,j})$, substituting the new values for the $\overline{T}^{y,t}(a_{i,j})$ on the right-hand side of the equations for the $\overline{T}^{x,t}(b_{i,j})$, doing one Gauss-Seidel iteration on those equations, substituting the new values for the $\overline{T}^{x,t}(b_{i,j})$ on the right-hand sides of the equations for the $\overline{T}^{y,t}(a_{i,j})$, and repeating the iterations until convergence was achieved. A similar iterative scheme was used to obtain the numerical solutions via the hybrid FD/NIM.

4.1 The Temporal Errors

To establish the order of the temporal error empirically for each method, the first test problem, the one with exact solution $T(x,y,t) = T_o + T_o t^4$, was solved using a 10×5 spatial grid, corresponding to 0.2×0.2 spatial cell size. The size of the time step was taken as $t_n - t_{n-1} = \Delta t =$ 0.6, and the first calculation using the FD/NIM was run. Then the time step was halved, and the second calculation was run. This procedure was repeated until a total of seven calculations were completed, the final one done using a time step of 0.009375; and the percent average L_1 error was calculated and tabulated at time $t =$ 0.60. The same procedure was used for the calculations done via the full space-time NIM. Both sets of calculations were done for Peclet Numbers, Pe = 1, 10, 10^2, 10^3, 10^6 and 10^{20}. The resulting errors for the two methods at $t =$ 0.60 are shown in Table 1, in which the orders of the errors also are tabulated, the latter being based simply on the difference of the errors and the time step sizes in two successive calculations.

Table 1 Comparison of the orders of the temporal errors for the 2-D hybrid FD/NIM and the 2-D full space-time NIM. An iterative convergence criterion of 10^{-14} was used in all calculations.

a) Pe = 1

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Er.	Full NIM Percent Avg. L-1 Rel. Er.	Ord. of FD/ NIM Temp. Er.	Ord. of Full NIM Temp. Er.
0.6	0.2	2.934205E+00	5.820030E+00		
0.3	0.2	1.934425E+00	2.276577E+00	0.60106	1.35416
0.15	0.2	1.113464E+00	5.882202E-01	0.79685	1.95244
0.075	0.2	5.978687E-01	1.484917E-01	0.89715	1.98597
0.0375	0.2	3.098085E-01	3.722491E-02	0.94845	1.99604
0.01875	0.2	1.576383E-01	9.312775E-03	0.97476	1.99899
0.009375	0.2	7.944600E-02	2.328603E-03	0.98857	1.99975

b) Pe = 10

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Er.	Full NIM Percent Avg. L-1 Rel. Er.	Ord. of FD/ NIM Temp. Er.	Ord. of Full NIM Temp. Er.
0.6	0.2	1.329711E+01	3.112438E+00		
0.3	0.2	7.204598E+00	1.056751E+00	0.88412	1.55841
0.15	0.2	3.685002E+00	2.758744E-01	0.96725	1.93755
0.075	0.2	1.851080E+00	6.976954E-02	0.99330	1.98334
0.0375	0.2	9.244319E-01	1.749275E-02	1.00173	1.99584
0.01875	0.2	4.604619E-01	4.376336E-03	1.00549	1.99896
0.009375	0.2	2.289910E-01	1.094281E-03	1.00779	1.99974

c) Pe = 10^2

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Er.	Full NIM Percent Avg. L-1 Rel. Er.	Ord. of FD/ NIM Temp. Er.	Ord. of Full NIM Temp. Er.
0.6	0.2	2.105499E+01	1.536865E+00		
0.3	0.2	1.047547E+01	4.744823E-01	1.00715	1.69556
0.15	0.2	5.077600E+00	1.257228E-01	1.04480	1.91611
0.075	0.2	2.469311E+00	3.186988E-02	1.04004	1.97998
0.0375	0.2	1.208339E+00	7.998367E-03	1.03108	1.99442
0.01875	0.2	5.937234E-01	2.001701E-03	1.02516	1.99848
0.009375	0.2	2.925239E-01	5.005573E-04	1.02124	1.99962

d) Pe = 10^3

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Er.	Full NIM Percent Avg. L-1 Rel. Er.	Ord. of FD/ NIM Temp. Er.	Ord. of Full NIM Temp. Er.
0.6	0.2	2.328655E+01	1.452786E+00		
0.3	0.2	1.136386E+01	4.991012E-01	1.03504	1.54142
0.15	0.2	5.457209E+00	1.372814E-01	1.05822	1.86220
0.075	0.2	2.646612E+00	3.485719E-02	1.04402	1.97761
0.0375	0.2	1.295666E+00	8.744443E-03	1.03045	1.99502
0.01875	0.2	6.373861E-01	2.188192E-03	1.02345	1.99863
0.009375	0.2	3.140851E-01	5.471784E-04	1.02101	1.99966

e) Pe = 10^6

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Er.	Full NIM Percent Avg. L-1 Rel. Er.	Ord. of FD/ NIM Temp. Er.	Ord. of Full NIM Temp. Er.
0.6	0.2	2.366517E+01	1.540743E+00		
0.3	0.2	1.153048E+01	5.325997E-01	1.03731	1.53250
0.15	0.2	5.540919E+00	1.460920E-01	1.05726	1.86617
0.075	0.2	2.695179E+00	3.717056E-02	1.03974	1.97465
0.0375	0.2	1.326454E+00	9.327229E-03	1.02281	1.99464
0.01875	0.2	6.576160E-01	2.334056E-03	1.01226	1.99861
0.009375	0.2	3.273421E-01	5.836554E-04	1.00645	1.99965

f) Pe = 10^{20}

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Er.	Full NIM Percent Avg. L-1 Rel. Er.	Ord. of FD/ NIM Temp. Er.	Ord. of Full NIM Temp. Er.
0.6	0.2	2.366562E+01	1.540865E+00		
0.3	0.2	1.153070E+01	5.326448E-01	1.03731	1.53249
0.15	0.2	5.541052E+00	1.461040E-01	1.05725	1.86618
0.075	0.2	2.695275E+00	3.717363E-02	1.03973	1.97464
0.0375	0.2	1.326534E+00	9.327997E-03	1.02277	1.99464
0.01875	0.2	6.576875E-01	2.334248E-03	1.01219	1.99861
0.009375	0.2	3.274087E-01	5.837034E-04	1.00631	1.99965

The results in Table 1 clearly show that, as expected based on their formulations in Sec. 3 and Sec. 2, the hybrid FD/NIM is first order in time, and the full space-time NIM is second order in time. For Pe = 1, the errors that result from the hybrid FD/NIM for the two largest time step sizes are smaller than the corresponding errors to which the full space-time NIM leads. However, this is the case only for these two extremely large time step sizes and this very low, and least interesting, Peclet number. Moreover, while the hybrid FD/NIM errors for these two largest time step sizes increase significantly as the Peclet number is increased, the full space-time NIM errors for those larger time step sizes remain small and those for the full space-time NIM are less than the corresponding errors for the hybrid FD/NIM for all other time step sizes and Pe studied. While it is clear that the asymptotic first and second order errors in the time step size for the hybrid FD/NIM and the full space-time NIM have not been attained for the larger time step sizes, especially at low Peclet numbers, it also is clear that they have been attained for the smaller time step sizes and that they verify both the first order convergence of the hybrid FD/NIM and the second order convergence of the full space-time NIM.

4.2 The Spatial Errors

The asymptotic order of the error as a function of the spatial cell size was studied for both methods for the same wide-range of Peclet numbers (1 to 10^{20}). The results of the calculations are tabulated in Table 2. The solutions were computed at t = 0.8, and since the solution is time-independent, only one time-step was used for all calculations. The spatial mesh was successively doubled from 10×5 to 160×80. The results in Table 2 clearly show that both methods are second order in space for the whole range of Peclet numbers studied. They also show that for this test problem the full space-time NIM leads to smaller percent average L_1 relative errors than the hybrid FD/NIM for all Peclet numbers studied except Pe = 1, which of course is the least interesting case and the least challenging case as a test of numerical methods.

Table 2 Comparison of the orders of the spatial errors for the 2-D hybrid FD/NIM and the 2-D full space-time NIM. An iterative convergence criterion of 10^{-14} was used in all calculations.

a) Pe = 1

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Error	Full NIM Percent Avg. L-1 Rel. Error	Ord. of FD/ NIM Spat. Er.	Ord. of Full NIM Spat. Er.
0.8	0.2	1.126288E-01	5.181105E-01		
0.8	0.1	2.942731E-02	1.322499E-01	1.93635	1.96999
0.8	0.05	7.440115E-03	3.323011E-02	1.98376	1.99270
0.8	0.025	1.865203E-03	8.317950E-03	1.99599	1.99819
0.8	0.0125	4.666195E-04	2.080138E-03	1.99901	1.99955

b) Pe = 10

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Error	Full NIM Percent Avg. L-1 Rel. Error	Ord. of FD/ NIM Spat. Er.	Ord. of Full NIM Spat. Er.
0.8	0.2	7.765528E-01	5.202853E-01		
0.8	0.1	1.937437E-01	1.298120E-01	2.00293	2.00288
0.8	0.05	4.840604E-02	3.243539E-02	2.00089	2.00078
0.8	0.025	1.209988E-02	8.109730E-03	2.00019	1.99984
0.8	0.0125	3.024885E-03	2.027501E-03	2.00004	1.99995

c) Pe = 10^2

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Error	Full NIM Percent Avg. L-1 Rel. Error	Ord. of FD/ NIM Spat. Er.	Ord. of Full NIM Spat. Er.
0.8	0.2	1.254828E+00	1.020471E+00		
0.8	0.1	3.109705E-01	2.434955E-01	2.01264	2.06727
0.8	0.05	7.559471E-02	5.948545E-02	2.04042	2.03329
0.8	0.025	1.855337E-02	1.472412E-02	2.02660	2.01436
0.8	0.0125	4.605517E-03	3.668192E-03	2.01025	2.00504

d) Pe = 10^3

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Error	Full NIM Percent Avg. L-1 Rel. Error	Ord. of FD/ NIM Spat. Er.	Ord. of Full NIM Spat. Er.
0.8	0.2	1.491722E+00	1.194888E+00		
0.8	0.1	4.040092E-01	2.947583E-01	1.88452	2.01927
0.8	0.05	9.810325E-02	7.168980E-02	2.04202	2.03969
0.8	0.025	2.331402E-02	1.742563E-02	2.07310	2.04056
0.8	0.0125	5.483942E-03	4.239848E-03	2.08791	2.03913

e) Pe = 10^6

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Error	Full NIM Percent Avg. L-1 Rel. Error	Ord. of FD/ NIM Spat. Er.	Ord. of Full NIM Spat. Er.
0.8	0.2	1.540894E+00	1.224789E+00		
0.8	0.1	4.609437E-01	3.100758E-01	1.74110	1.98184
0.8	0.05	1.338060E-01	7.781194E-02	1.78445	1.99456
0.8	0.025	3.773834E-02	1.947467E-02	1.82604	1.99839
0.8	0.0125	1.041310E-02	4.868795E-03	1.85763	1.99996

f) Pe = 10^{20}

Time Step Size	Spatial Cell Size	FD/NIM Percent Avg. L-1 Rel. Error	Full NIM Percent Avg. L-1 Rel. Error	Ord. of FD/ NIM Spat. Er.	Ord. of Full NIM Spat. Er.
0.8	0.2	1.540943E+00	1.224820E+00		
0.8	0.1	4.610253E-01	3.100935E-01	1.74090	1.98179
0.8	0.05	1.339082E-01	7.782190E-02	1.78360	1.99446
0.8	0.025	3.786198E-02	1.948047E-02	1.82242	1.99815
0.8	0.0125	1.054702E-02	4.871935E-03	1.84391	1.99946

5. CONCLUSIONS

Two methods for the numerical solution of the time-dependent, two-dimensional, linear, convection-diffusion equation have been studied. The first, a full space-time NIM in which both the temporal and spatial operators are discretized using the nodal integral approach, is an extension to two-dimensional space-time problems of previously developed nodal integral methods for two-dimensional steady-state problems (Esser, 1993), (Michael, 1993) and for one-dimensional space-time problems (Rizwan-uddin, 1997). The second, a hybrid FD/NIM in which the temporal operator is discretized using a backward finite-difference approximation and then the spatial operator along with the terms that result from the finite-difference approximation of the time derivative are discretized using the nodal integral approach, differs from a previously developed hybrid FD/NIM (Elnawawy, 1990) in the ways the terms that result from the finite-difference approximation of the time derivative are treated in the subsequent development of the nodal integral method in space. The orders of the errors resulting from the two methods were calculated and compared. These comparisons showed that the full space-time NIM leads to second order errors in both space and time, and that the hybrid FD/NIM leads to a second order error in space, but only a first order error in time. Finally, they also showed that for all Peclet numbers studied except the lowest ($Pe = 1$), which is of the least practical and computational interest, the full space-time NIM led to smaller errors than the hybrid FD/NIM for the same time step size and spatial cell size. Thus to achieve comparable accuracy, the hybrid FD/NIM would have to be run using a smaller time step size and/or a smaller spatial cell size. Since the two methods required approximately the same amounts of CPU time when they were run using the same time step size and spatial cell size, the hybrid method requires considerably more CPU time than the full space-time NIM to achieve comparable accuracy. (See Tables 1 and 2.) It was on this basis, and on the not unrelated basis of the second order temporal convergence of the full space-time NIM, that the decision was made to develop the primitive-variable NIM for the Navier-Stokes equations (Michael, 2001a) using the principal ideas on which the full space-time NIM is based rather than those on which the hybrid FD/NIM is based.

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