Abstract

Key words:

1 Introduction

2 The process of angular-spatial transport in multiple scattering of charged particles

$\hat{\mathbf{x}}$ $\hat{\mathbf{y}}$ $\hat{\mathbf{z}}$

O $r = (x, y, z) \in \mathbf{R}^3$

$\Im = (p, r) = \mathbf{R}^3 \times \mathbf{R}^3$

$p = (p_x, p_y, p_z) \in \mathbf{R}^3$ $r = (x, y, z) \in \mathbf{R}^3$

$|p| = \sqrt{p_x^2 + p_y^2 + p_z^2}$

p u

p

$$p = |p|\,(\theta_x, \theta_y, \theta_z) = |p|\,u$$

u S $\mathbf{R}^3$
θ_x, θ_y θ_z
$\Im$
S $\mathbf{R}^3$
$\Im = \{(u, r) \equiv u \in S,\ r \in \mathbf{R}^3\}$
dr dt
u $dr = u dt$ u
S $u(t)$
0
t $r(t)$
$[0, t]$ $r(t)$
$r(t) = \int_0^t u(t')dt'$ $\int_0^t u(t')dt'$
$u(t')$

3 Definition of the evolution process of particle's directions

O
u_0 $\widehat{\mathbf{z}}$ $u(0) = u_0$ $r(0) = O$
$(u(t), r(t))$
$u(t)$ $p(t)$
$u(t)$

$u(t)$ 0 u_0
D
t $f_{diff}(t, u|u_0) \equiv f_{diff}(t, u - u_0) = f_{diff}(t, u)$ $f_{diff}(t, u)$

$$\frac{\partial f_{diff}(t, u)}{\partial t} = \frac{D}{2}\Delta_u f_{diff}(t, u)$$

$$f_{diff}(0,u) = \delta(u - u_0)$$

Δ_u S

τ $u(t)$

Δu

τ Δu

$u(t)$ τ

τ $u(t)$

$u(\tau) + \Delta u$ 0 u_0

τ $u(\tau)$ $\Delta u.$

τ

$(1/\lambda)\exp(-\lambda t)$

Δu

$\frac{1}{\sigma}\frac{d\sigma}{du}$

$p(u)$ $p(u) = \frac{1}{\sigma}\frac{d\sigma}{du}(u)$

S

$\frac{1}{\sigma}\frac{d\sigma}{du}$

$n(t)$

$u(t)$ $[0,t]$ λt

λ $u(t)$

$[0,1]$ $n(t)$

$\Pr\{\cdot\}$

dt

$$\begin{aligned}
\Pr\{n(dt) &= 0\} = 1 - \lambda dt + o(dt)\\
\Pr\{n(dt) &= 1\} = \lambda dt + o(dt)\\
\Pr\{n(dt) &\geq 2\} = o(dt)
\end{aligned}$$

$\Pr\{n(dt) = k\}$ k

dt $n(t)$

$(u(t), r(t))$

4 Transport equation

$(u(t), r(t))$

$(u, r) \in S \times \mathbf{R}^3$ t

$f(t, u, r)$

$(u(t), r(t))$ $(u, r) \in S \times \mathbf{R}^3$ t

0 (u_0, O) $f(t, u, r \mid 0, u_0, O)$

ΔV $S \times \mathbf{R}^3$ $\Pr((u(t+dt), r(t+dt)) \in \Delta V)$

ΔV

$t + dt$ $n(dt) = k, k = 0, 1, 2, \ldots.$

k $k = 0, 1, 2, \ldots.$

$(t, t + dt)$

$\Pr((u(t+dt), r(t+dt)) \in \Delta V)$

$$\begin{gathered}
\Pr((u(t+dt), r(t+dt)) \in \Delta V) = \\
\sum_{k=1}^{\infty} \Pr((u(t+dt), r(t+dt)) \in \Delta V \;\; and \; n(dt) = k) = \\
\sum_{k=1}^{\infty} \Pr((u(t+dt), r(t+dt)) \in \Delta V \mid n(dt) = k) \Pr(n(dt) = k) = \\
\Pr((u(t+dt), r(t+dt)) \in \Delta V \mid n(dt) = 0) \Pr(n(dt) = 0) + \\
\Pr((u(t+dt), r(t+dt)) \in \Delta V \mid n(dt) = 1) \Pr(n(dt) = 1) + o(dt) = \\
\Pr((u(t+dt), r(t+dt)) \in \Delta V \mid n(dt) = 0)(1 - \lambda dt) + \\
\Pr((u(t+dt), r(t+dt)) \in \Delta V \mid n(dt) = 1)\lambda dt + o(dt)
\end{gathered}$$

$\Pr((u(t+dt), r(t+dt)) \in \Delta V \mid n(t) = 0)$

$((u(t), r(t))$

$(t, t + dt)$ ΔV

$t + dt$

$$\begin{gathered}
\Pr\{(r(t+dt), u(t+dt)) \in \Delta V \quad \mid \quad n(dt) = 0\} = \\
\int_{\Delta V} \int_{S} f(t, u - u'', r - dt \cdot (u - u'') + o(dt)) f_{diff}(dt, u'') du'' du dr
\end{gathered}$$

$u - u'' = u'$ t

$r - dt \cdot (u - u'') + o(dt)$ t $f_{diff}(dt, u'')$

u''

$(t, t + dt)$

$f(t, u - u'', r - dt \cdot (u - u'') + o(dt))$

u'' $dt \cdot (u - u'')$

$$\int_{\Delta V}\int_{S} f(t,u-u'',r-dt\cdot(u-u'')+o(dt))f_{diff}(dt,u'')du''dudr =$$

$$\int_{\Delta V}\int_{S}\{f(t,u,r)-\nabla_u f(t,u,r)(u'')-\nabla_r f(t,u,r)(u-u'')dt$$

$$+\tfrac{1}{2}\Delta_u f(t,u,r)(u'')^2+o(dt)\}f_{diff}(dt,u'')du''dudr =$$

$$\int_{\Delta V}\{f(t,u,r)+\nabla_r f(t,u,r)\cdot udt+\tfrac{1}{2}(D\cdot dt)\Delta_u f(t,u,r)\}dudr+o(dt).$$

∇_r r $\nabla_r = (\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$ Δ_u S $f_{diff}(dt,u'')$ $u(t)$ $(t,t+dt)$

$$\int_S (u'')f_{diff}(dt,u'')du'' = o(dt),$$

$$\int_S ((u-u'')dt)f_{diff}(u'';dt)du'' = udt+o(dt)$$

$$and \quad \int_S (u'')^2 f_{diff}(dt,u'')du'' = (D\cdot dt)+o(dt)$$

u'',r $u''dt$ $o(dt)$

$$\Pr\{(r(t+dt),u(t+dt))\in\Delta V \mid n(dt)=1\}$$

$((u(t),r(t))$ $(t,t+dt)$ ΔV $t+dt$ u'' t t' t t' $t+dt$ $\widetilde{u}$ t' u''' t' $t+dt$ u'', $t',\widetilde{u}$ u''' $(u'',t',\widetilde{u},u''')$

$$\frac{f_{diff}(t'-t,u'')\ p(\widetilde{u})f_{diff}(t+dt-t',u''')}{dt}$$

$$\Pr\{(r(t+dt),u(t+dt))\in\Delta V \mid n_{hard}(dt)=1\}$$

$$\Pr\{(r(t+dt),u(t+dt))\in\Delta V \mid n_{hard}(dt)=1\} =$$

$$\int_{\Delta V}\int_S\int_S\int_S\int_{(t,t+dt)} f(t,u-u'''-\widetilde{u}-u'',r-(t'-t)\cdot u'-(t+dt-t')\cdot(u-u''')+o(dt))$$

$$f_{diff}(t'-t,u'')p(\widetilde{u})f_{diff}(t+dt-t',u''')\tfrac{1}{dt}dt'du'''d\widetilde{u}du''dudr$$

$(t, t+dt)$

$f(t, u - u''' - \widetilde{u}, r - dt \cdot (u - u''') + o(dt))$

u''' $dt \cdot (u - u''')$

$f_{diff}(dt, u'')$

$$\int\limits_{\Delta V} \int\limits_{S} \int\limits_{S} \int\limits_{S} \int\limits_{(t,t+dt)} f(t, u - u''' - \widetilde{u} - u'', r - (t'-t)\cdot u' - (t+dt-t')\cdot(u-u''') + o(dt))$$
$$f_{diff}(t'-t, u'') p(\widetilde{u}) f_{diff}(t+dt-t', u''') \frac{1}{dt} dt' du''' d\widetilde{u} du'' du dr =$$
$$\int\limits_{\Delta V} \int\limits_{S} \int\limits_{S} f(t, u - u''' - \widetilde{u}, r - dt \cdot (u - u''') + o(dt))$$
$$p(\widetilde{u}'') f_{diff}(dt, u''') du''' d\widetilde{u} du dr =$$
$$\int\limits_{\Delta V} \int\limits_{S} \int\limits_{S} \{ f(t, u - \widetilde{u}, r) - \nabla_r f(t, u - \widetilde{u}, r)(u - u''') dt$$
$$+ \frac{1}{2} \Delta_u f(t, u - \widetilde{u}, r - dt \cdot (u - u'''))(u''')^2 + o(dt) \}$$
$$p(\widetilde{u}) f_{diff}(dt, u''') du''' d\widetilde{u} du dr + o(dt) =$$
$$\int\limits_{\Delta V} \int\limits_{S} f(t, u - \widetilde{u}, r) p(\widetilde{u}) d\widetilde{u} du dr +$$
$$+ dt \int\limits_{\Delta V} \int\limits_{S} \{ -\nabla_r f(t, u - \widetilde{u}, r) \cdot u + \frac{D}{2} \Delta f(t, u - \widetilde{u}, r) \} p(\widetilde{u}) d\widetilde{u} du dr + o(dt).$$

$$\Pr\{(r(t+dt), u(t+dt)) \in \Delta V\} = \int\limits_{\Delta V} f(t+dt, u, r) du dr =$$
$$(1 - \lambda dt) \int\limits_{\Delta V} \{ f(t, u, r) - \nabla_r f(t, u, r) \cdot u dt + \frac{D}{2} \Delta_u f(t, u, r) dt \} du dr +$$
$$\lambda dt \int\limits_{\Delta V} \int\limits_{S} f(t, u - \widetilde{u}, r) p(\widetilde{u}) d\widetilde{u} du dr + o(dt) =$$
$$\int\limits_{\Delta V} f(t, u, r) du dr - \lambda dt \int\limits_{\Delta V} f(t, u, r) du dr - dt \int\limits_{\Delta V} \nabla_r f(t, u, r) \cdot u du dr +$$
$$+ \frac{1}{2} D \cdot dt \int\limits_{\Delta V} \Delta_u f(t, u, r) du dr + \lambda dt \int\limits_{\Delta V} \int\limits_{S} f(t, u - \widetilde{u}, r) p(\widetilde{u}) d\widetilde{u} du dr + o(dt).$$

$$\int\limits_{\Delta V} \left(f(t+dt, u, r) - f(t, u, r) \right) du dr = -dt \int\limits_{\Delta V} \nabla_r f(t, u, r) \cdot u \; du dr$$
$$+ \frac{1}{2} D \cdot dt \int\limits_{\Delta V} \Delta_u f(t, u, r) du dr + \lambda dt \int\limits_{\Delta V} \int\limits_{S} \left(f(t, u - \widetilde{u}, r) - f(t, u, r) \right) p(\widetilde{u}) d\widetilde{u} du dr + o(dt).$$

$dt \to 0,$ ΔV $\Delta V \to 0$

$$\frac{\partial f(t,u,r)}{\partial t} + u \cdot \nabla_r f(t, u, r) = \frac{D}{2} \Delta_u f(t, u, r) + \lambda \int\limits_{S} \left[f(t, u - \widetilde{u}, r) - f(t, u, r) \right] p(\widetilde{u}) d\widetilde{u}$$

$f(t, u, r)$

$$f(t,u,r)|_{t=0} = f(0,u,r \mid 0,u_0,O) = \delta(u-u_0)\delta(r-O)$$

[illegible] D [illegible] 0 [illegible] $u(t)$ [illegible]

$$\frac{\partial f(t,u,r)}{\partial t} + u \cdot \nabla_r f(t,u,r) = \lambda \int_S \left[f(t,u-\widetilde{u},r) - f(t,u,r) \right] p(\widetilde{u}) d\widetilde{u}$$

[illegible] λ [illegible] 0 [illegible] $u(t)$ [illegible] $u(t)$ [illegible] D [illegible] $f_{diff}(t,u,r)$ [illegible] $(u(t),r(t))$ [illegible] $r(t)$ [illegible] $r(t) = \int_0^t u(t')dt'$ [illegible]

$$\frac{\partial f_{diff}(t,u,r)}{\partial t} + u \cdot \nabla_r f_{diff}(t,u,r) = \frac{D}{2} \Delta_u f_{diff}(t,u,r)$$

5 Solution

[illegible] $n(t)$ [illegible] $S \times \mathbf{R}^3$ [illegible] $[0,t]$ [illegible] $f(t,u,r \mid 0,u_0,O)$ [illegible]

$$f(t,u,r \mid 0,u_0,O) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} p(t,u,r \mid 0,u_0,O,n(t)=n)$$

$$p(t,u,r \mid 0,u_0,O,n(t)=k) = p^{(n)}(t,u,r \mid 0,u_0,O)$$

[illegible] $((u(t),r(t))$ [illegible] $S \times \mathbf{R}^3$ [illegible] (u_0,O) [illegible] (u,r) [illegible] $[0,t)$ [illegible] k [illegible] $[0,t)$ [illegible] $p^{(n)}(t,u,r \mid 0,u_0,O)$ [illegible] $n = 0,1,2,..$ [illegible] $p^{(n)}(t,u,r \mid 0,u_0,O)$ [illegible] $\Pr((u(t),r(t)) \in \Delta V \mid 0,u_0,O;n(t)=n)$ [illegible] ΔV [illegible] $S \times \mathbf{R}^3$ [illegible] t

[illegible]

u_0, O 0 n 0 t

$$\Pr((u(t), r(t)) \in \Delta V \mid 0, u_0, O; n(t) = n)$$

$p^{(n)}(t, u, r \mid 0, u_0, O)$ u, r $\Delta V \subset S \times \mathbf{R}^3$

$$\Pr((u(t), r(t)) \in \Delta V \mid 0, u_0, O; n(t) = n) = \int_{\Delta V} p^{(n)}(t, u, r \mid 0, u_0, O)\, du dr.$$

$[0, t]$ N

$$I_1 = [0, t/N), I_2 = [t/N, 2t/N), \dots I_N = [(N-1)t/N, t].$$

$n - th$

$t_1, t_2, \dots, t_n \; (t_1 < t_2 < \dots t_n)$

$$\Pr((u(t), r(t))\Delta V \mid 0, u_0, O; n(t) = n) =$$

$$\sum_{k=1}^{N} \Pr((u(t), r(t)) \in \Delta V \; ; t_{n-1} < (k-1)t/N, \; t_n \in I_k \mid 0, u_0, O; n(t) = n) + o(1/N) =$$

$$\sum_{k=1}^{N} \Pr(t_{n-1} < (k-1)t/N, t_n \in I_k \mid n(t) = n) \cdot$$

$$\Pr((u(t), r(t)) \in \Delta V \mid 0, u_0, O; \; t_{n-1} < (k-1)t/N, t_n \in I_k, n(t) = n) + o(1/N).$$

$k = 1, 2, \dots N$ $k = 1$

$k = 1$ 0 $(t_1, t_2, \dots t_n) \in I_1$

$o(1/N)$

$\Pr(t_{n-1} < (k-1)t/N, t_n \in I_k \mid n(t) = n)$

$$\Pr(t_{n-1} < (k-1)t/N, t_n \in I_k \mid n(t) = n) = \frac{(k-1)^{n-1} \cdot n}{N^n} + O(1/N)$$

ΔV

$(n-1)$ $[0, (k-1)t/N]$ I_k

n $[0, t)$

$u(t)$ $(u(t), r(t)$ I_k $(t_{n-1} < (k-1)t/N, t_n \in I_k, n(t) = n)$

(u_1, r_1) (u', r')

$[(k-1)t/N, t']$ $t' \in I_k$

$(t', \widetilde{u}) \in I_k \times S$ t'

$(u' + \widetilde{u}, r')$ (u, r)

$[t', kt/N]$ (u, r)

t' $n - th$ $\widetilde{u}$

$\widetilde{u}$ t'

$(u(t), r(t))$

I_k $G = S \times \mathbf{R}^3 \times S \times \mathbf{R}^3 \times [(k-1)t/N, kt/N] \times S$ G

$$f_{diff}(t', u', r' \mid (k-1)t/N, u_1, r_1) \cdot \frac{N}{t} \cdot p(\widetilde{u}) \cdot$$
$$f_{diff}(t, u, r \mid t', u' + \widetilde{u}, r') dr_1 du_1 dr' du' dt' d\widetilde{u}.$$

$f_{diff}(t, u, r \mid t_1, u_1, r_1)$ $(u(t), r(t))$ t_1 (u_1, r_1) t (u, r) $u(t)$ D $r(t)$ $r(t) = \int_0^t u(t')dt'$ $\lambda = 0$ $f_{diff}(t, u, r \mid t_1, u_1, r_1)$

$$\frac{\partial f_{diff}(t, u, r)}{\partial t} + u \cdot \nabla_r f_{diff}(t, u, r) = \frac{D}{2} \Delta_u f_{diff}(t, u, r)$$

$$f_{diff}(t, u, r)|_{t=t_1} = f(t_1, u, r \mid t_1, u_1, r_1) = \delta(u - u_1)\delta(r - r_1)$$

$$\Pr((u(t), r(t)) \in \Delta V \mid 0, u_0, O\ ;\ t_{n-1} < (k-1)t/N, t_n \in I_k, n(t) = n) =$$
$$\int\limits_{G \times \Delta V} p^{(n-1)}((k-1)t/N, u_1, r_1 \mid 0, u_0, O) f_{diff}(t', u', r' \mid (k-1)t/N, u_1, r_1)$$
$$\tfrac{N}{t} \cdot p(\widetilde{u}) f_{diff}(t, u, r \mid t', u' + \widetilde{u}, r') dr_1 du_1 dr' du' dt' d\widetilde{u} du dr$$

t'

$$\Pr((u(t), r(t)) \in \Delta V \mid 0, u_0, O\ ;\ t_{n-1} < (k-1)t/N, t_n \in I_k, n(t) = n) =$$
$$\{ \int\limits_{S \times \mathbf{R}^3 \times S \times \mathbf{R}^3 \times \Delta V} p^{(n-1)}((k-1)t/N, u_1, r_1 \mid 0, u_0, O)$$
$$f_{diff}(kt/N, u', r' \mid (k-1)t/N, u_1, r_1)$$
$$\left[\int\limits_S p(\widetilde{u}) f_{diff}(t, u, r \mid kt/N, u' + \widetilde{u}, r') d\widetilde{u} \right] du_1 dr_1 du' dr' \} du dr + O(\tfrac{\Delta V}{N})$$

$$\int\limits_S p(\widetilde{u}) f_{diff}(t, u, r \mid kt/N, u' + \widetilde{u}, r') d\widetilde{u}$$

$(kt/N, u', r')$ (t, u, r)

[illegible] kt/N [illegible] f_{diff} [illegible] p [illegible] $\widetilde{f}_{diff}(t,u,r \mid kt/N,u',r')$

$$\widetilde{f}_{diff}(t,u,r \mid kt/N,u',r') = \int\limits_{S} p_{hard}(\widetilde{u}) f_{diff}(t,u,r \mid kt/N,u'+\widetilde{u},r')d\widetilde{u}$$

[illegible]

$$\begin{aligned}&\Pr((u(t),r(t)) \in \Delta V \mid 0,u_0,O; n(t)=n) = \\ &\sum_{k=1}^{N} \tfrac{(k-1)^{n-1}\cdot n}{N^n} \int\limits_{S\times\mathbf{R}^3\times S\times\mathbf{R}^3\times\Delta V} p^{(n-1)}((k-1)t/N,u_1,r_1 \mid 0,u_0,O) \\ &f_{diff}(kt/N,u',r' \mid (k-1)t/N,u_1,r_1) \\ &\widetilde{f}_{diff}(t,u,r \mid kt/N,u',r')du_1dr_1du'dr'dudr + O(\Delta V/N)\end{aligned}$$

[illegible] $N \to \infty$ [illegible] $f_{diff}(kt/N,u',r' \mid (k-1)t/N,u_1,r_1)$ [illegible] $u'-u_1$ [illegible] $r'-r_1$ [illegible] $\sum_{k=1}^{N} \frac{(k-1)^{n-1}\cdot n}{N^n}$ [illegible] $\int_0^t \ldots \tau^{n-1}d\tau$ [illegible] $N \to \infty$ [illegible] $du'dr'$ [illegible] $S\times\mathbf{R}^3$ [illegible] ΔV [illegible] $\Delta V \to 0$ [illegible]

$$\begin{aligned}p^{(n)}(t,u,r \mid 0,u_0,O) = \frac{n}{t^n}\int\limits_{S\times\mathbf{R}^3}[\int\limits_0^t p^{(n-1)}(\tau,u_1,r_1 \mid 0,u_0,O) \\ \widetilde{f}_{diff}(t,u,r \mid \tau,u_1,r_1)\tau^{n-1}d\tau]du_1dr_1\end{aligned}$$

[illegible] $p^{(n)}(t,u,r \mid 0,u_0,O)$ [illegible] $f_{diff}(t,u,r \mid t',u_1,r_1)$ [illegible] $f_{diff}(t,u,r \mid t',u_1,r_1)$ [illegible] $f_{diff}(t,u,r \mid t',u_1,r_1)$ [illegible]

6 The small angle approximation for diffusional transport

6.1 First order approximation in angular and spatial variables

$$T_{sa} \quad : \quad S \to P, \; T_{sa}(u) = A$$
$$where \quad u \; = \; (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \quad A = (\theta\cos\varphi, \theta\sin\varphi)$$

$$\frac{\partial f_{diff}(t,A,r)}{\partial t} + A \cdot \nabla_r f(t,A,r) = \frac{D}{2} \Delta f_{diff}(t,A,r)$$

[illegible] t [illegible] $f_{diff}(t, A, r \mid t', A_1, r_1)$ [illegible] $f_{diff}(t,\theta_x,\theta_y,x,y,t \mid t',\theta_x^1,\theta_y^1,x_1,y_1,t')$ [illegible] t [illegible] z t' [illegible] z', (θ_x^1,θ_y^1) [illegible] A_1 [illegible] (x_1,y_1,t') [illegible] r_1 [illegible] **??** [illegible] $f_F(t,\theta_x,\theta_y,x,y,\mid t',\theta_x^1,\theta_y^1,x_1,y_1)$

$$f_F(t,\theta_x,\theta_y,x,y \mid t',\theta_x^1,\theta_y^1,x_1,y_1) = f_{diff}(t,\theta_x,\theta_y,x,y,t \mid t',\theta_x^1,\theta_y^1,x_1,y_1,t')$$

[illegible] $f_F(t,\theta_x,\theta_y,x,y,\mid t',\theta_x^1,\theta_y^1,x_1,y_1)$ [illegible] **??** [illegible]

$$\frac{\partial f_F}{\partial t} = -\theta_x\frac{\partial f_F}{\partial x} - \theta_y\frac{\partial f_F}{\partial y} + \frac{D}{2}\left(\frac{\partial^2 f_F}{\partial\theta_x^2} + \frac{\partial^2 f_F}{\partial\theta_y^2}\right)$$

[illegible]

$$f_F\mid_{t=t'} = \delta(\theta_x-\theta_x^1)\delta(\theta_y-\theta_y^1)\delta(x-x_1)\delta(y-y_1)$$

[illegible]

$$f_F(t,\theta_x,\theta_y,x,y \mid t',\theta_x^1,\theta_y^1,x_1,y_1) = f_F(t,\theta_x,x \mid t',\theta_x^1,x_1)f_F(t,y,\theta_y \mid t',\theta_y^1,y_1)$$

$$\begin{aligned} f_F(t,\theta_x,x \quad \mid \quad & t',\theta_x^1,x_1) = \\ & \frac{1}{2\pi\sigma_\theta(t,t')\sigma(t,t')\sqrt{1-r^2(t,t')}}\exp\{-\frac{1}{2(1-r^2(t,t'))} \\ & \left[\frac{(\theta_x-\theta_x^1)^2}{\sigma_\theta^2(t,t')} - 2r(t,t')\frac{(x-x_1-(t-t')\theta_x^1)(\theta_x-\theta_x^1)}{\sigma_\theta(t,t')\sigma(t,t')} + \frac{(x-x_1-(t-t')\theta_x^1)^2}{\sigma^2(t,t')}\right]\} \end{aligned}$$

$$\begin{aligned} f_F(t,\theta_y,y \quad \mid \quad & t',\theta_y^1,y_1) = \\ & \frac{1}{2\pi\sigma_\theta(t,t')\sigma(t,t')\sqrt{1-r^2(t,t')}}\exp\{-\frac{1}{2(1-r^2(t,t'))} \\ & \left[\frac{(\theta_y-\theta_y^1)^2}{\sigma_\theta^2(t,t')} - 2r(t,t')\frac{(y-y_1-(t-t')\theta_y^1)(\theta_y-\theta_y^1)}{\sigma_\theta(t,t')\sigma(t,t')} + \frac{(y-y_1-(t-t')\theta_y^1)^2}{\sigma^2(t,t')}\right]\} \end{aligned}$$

$$\begin{aligned} where \quad \sigma_\theta^2(t,t') &= A_0(t,t'), \sigma^2(t,t') = A_2(t,t'), \quad r(t,t') = \frac{A_1(t,t')}{\sqrt{A_0(t,t')A_2(t,t')}} \\ with \quad A_i(t,t') &= D\int_{t'}^{t}(t-t)^i dt, \qquad i = 0,1,2. \end{aligned}$$

[illegible]

$f_{diff}(t, u, r|t', u', r')$

$f_F(t, \theta_x, \theta_y, x, y \mid t', \theta_x^1, \theta_y^1, x_1, y_1)$

$$\begin{aligned} f_F(t, \theta_x, \theta_y, x, y \quad &| \quad t', \theta_x^1, \theta_y^1, x_1, y_1) \\ &= \quad f_{diff}(t, (\theta \cos\varphi, \theta \sin\varphi), (x, y)|t', (\theta_1 \cos\varphi_1, \theta_1 \sin\varphi_1), (x_1, y_1)) \\ &= \quad f_{diff}(t, \theta, \varphi, x, y, z = t|t', \theta_1, \varphi_1, x_1, y_1, z_1 = t') \\ &= \quad f_{diff}(t, u, r|t', u_1, r_1) \end{aligned}$$

f t (z, t) $z = t$ $f_{diff}(t, u, r \mid t', u_1, r_1)$ $f_F(t, \theta_x, \theta_y, x, y \mid t', \theta_x^1, \theta_y^1, x_1, y_1)($ $f_{diff}(t, u, r \mid t', u_1, r_1)$ $\widetilde{f}_{diff}(t, u, r \mid t', u_1, r_1)$

6.2 Yang approximation

$\widehat{z}$ θ $u = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in S$ $(\theta\cos\varphi, \theta\sin\varphi, 1 - \theta^2/2)$ $(\theta_x, \theta_y, \theta_z)$ Υ B $\theta_z = 1 - (\theta_x^2 + \theta_y^2)/2$ S $\Upsilon = (\theta_x, \theta_y, \theta_z) = (\theta\cos\varphi, \theta\sin\varphi, 1 - \theta^2/2)$ $u \cdot \nabla_r f_{diff}$

$$\theta_x \frac{\partial f_{diff}}{\partial x} + \theta_y \frac{\partial f_{diff}}{\partial y} + [1 - (\theta_x^2 + \theta_y^2)/2] \frac{\partial f_{diff}}{\partial z}$$

θ $u \cdot \nabla_r f_{diff}$ Δ_u θ Δ_u 2 $\Delta = (\frac{\partial^2 f}{\partial \theta_x^2} + \frac{\partial^2 f}{\partial \theta_y^2})$ P $u(t)$

P S

$r(t)$

$z-$

$$\frac{\partial f_{diff}}{\partial t} = -\theta_x \frac{\partial f_{diff}}{\partial x} - \theta_y \frac{\partial f_{diff}}{\partial y} - [1-(\theta_x^2+\theta_y^2)/2]\frac{\partial f_{diff}}{\partial z} + \frac{D}{2}(\frac{\partial^2 f_{diff}}{\partial \theta_x^2} + \frac{\partial^2 f_{diff}}{\partial \theta_y^2})$$

f_{diff} $(t,\Upsilon,r) = (t,\theta_x,\theta_y,1-(\theta_x^2+\theta_y^2)/2,x,y,z)$. $(t,\theta_x,\theta_y,x,y,\epsilon)$ $(t,\Upsilon,r) = (t,\theta_x,\theta_y,1-(\theta_x^2+\theta_y^2)/2,x,y,z)$ $\epsilon = (t-z)$ $f_Y(t,\theta_x,\theta_y,x,y,\epsilon)$ $f_Y(t,\theta_x,\theta_y,x,y,\epsilon) = f_{diff}(t,\theta_x,\theta_y,1-(\theta_x^2+\theta_y^2)/2,x,y,z)$

$$\frac{\partial f_Y}{\partial t} = -\theta_x \frac{\partial f_Y}{\partial x} - \theta_y \frac{\partial f_Y}{\partial y} - \frac{1}{2}(\theta_x^2+\theta_y^2)\frac{\partial f_Y}{\partial \epsilon} + \frac{D}{2}(\frac{\partial^2 f_Y}{\partial \theta_x^2} + \frac{\partial^2 f_Y}{\partial \theta_y^2})$$

$(\theta_x,\theta_y,x,y,\epsilon) = (0,0,0,0,0)$

$$f_Y(t,\theta_x,\theta_y,x,y,\epsilon)\mid_{t=0} = \delta(\theta_x)\delta(\theta_y)\delta(x)\delta(y)\delta(\epsilon)$$

$$\begin{aligned} f_Y(t,\theta_x,\theta_y,x,y,\epsilon) &= f_Y(t,\rho,\psi,\theta,\varphi,\epsilon) \\ &= \frac{\sqrt{2}[Dt^3+3\rho^2/2+t^2\theta^2/2-t\theta\rho\cos(\varphi-\psi)]^7}{5\sqrt{\pi^5 D^{11} t^{27}}(2t\epsilon-\rho^2)^{9/2}} \\ &\quad \cdot \exp\{-\frac{2t^2\theta^2+6\rho^2-6t\rho\theta\cos(\varphi-\psi)}{Dt^3}\} \\ &\quad \cdot \exp\{-\frac{[Dt^3+3\rho^2/2+t^2\theta^2/2-t\theta\rho\cos(\varphi-\psi)]^2}{2Dt^3(2t\epsilon-\rho^2)}\} \end{aligned}$$

$\rho,\psi,t-\epsilon$ $\mathbf{R}^3$ θ,φ $A =$ $(\theta\cos\varphi,\theta\sin\varphi) = (\theta_x,\theta_y)$ P θ_x,θ_y,x,y $\rho,\theta,\varphi-\psi$

$$\rho = \sqrt{x^2+y^2}, \quad \theta = \sqrt{\theta_x^2+\theta_y^2}, \qquad \varphi-\psi = \arccos\left(\frac{x\theta_x+y\theta_y}{\sqrt{x^2+y^2}\sqrt{\theta_x^2+\theta_y^2}}\right).$$

$f_Y(t,\theta_x,\theta_y,x,y,\epsilon)$

$f_Y(t,\theta_x,\theta_y,x,y,\epsilon \mid 0,0,0,0,0)$

$f_Y(t,\theta_x,\theta_y,x,y,\epsilon \mid t',\theta_x^1,\theta_y^1,x_1,y_1,\epsilon_1)$

t'

$(t',\theta_x^1,\theta_y^1,x_1,y_1,\epsilon_1)$

$$f_Y(t,\theta_x,\theta_y,x,y,\epsilon \mid t',\theta_x^1,\theta_y^1,x_1,y_1,\epsilon_1)|_{t=t'} = \delta(\theta_x-\theta_x^1)\delta(\theta_y-\theta_y^1)\delta(x-x_1)\delta(y-y_1)\delta(\epsilon-\epsilon_1)$$

$$f_Y(t,\theta_x,\theta_y,x,y,\epsilon \mid t',\theta_x^1,\theta_y^1,x_1,y_1,\epsilon_1) =$$
$$f_Y(t-t',\theta_x-\theta_x^1,\theta_y-\theta_y^1,x-x_1-[(t-t')-(\epsilon-\epsilon_1)]\theta_x^1,y-y_1-[(t-t')-(\epsilon-\epsilon_1)]\theta_y^1,\epsilon-\epsilon_1 \mid 0,0,0,0,0) =$$

$$f_Y(t-t',\theta_x-\theta_x^1,\theta_y-\theta_y^1,x-x_1-[(t-t')-(\epsilon-\epsilon_1)]\theta_x^1,y-y_1-[(t-t')-(\epsilon-\epsilon_1)]\theta_y^1,\epsilon-\epsilon_1)$$

$(u(t),r(t))$

$(t,\theta_x,\theta_y,x,y,\epsilon)$ $(t-t',\theta_x-\theta_x^1,\theta_y-\theta_y^1,x-x_1-[(t-t')-(\epsilon-\epsilon_1)]\theta_x^1,y-y_1-[(t-t')-(\epsilon-\epsilon_1)]\theta_y^1,\epsilon-\epsilon_1))$

$f_{diff}(t,u,r|t',u',r')$

$f_Y(t,\theta_x,\theta_y,x,y,\epsilon \mid t',\theta_x^1,\theta_y^1,x_1,y_1,\epsilon_1)$

$$\begin{aligned}
f_Y(t,\theta_x,\theta_y,x,y,\epsilon \quad &\mid \quad t',\theta_x^1,\theta_y^1,x_1,y_1,\epsilon_1) \\
&= f_Y(t,(\theta\cos\varphi,\theta\sin\varphi),(x,y),\epsilon \mid t',(\theta_1\cos\varphi_1,\theta_1\sin\varphi_1),(x_1,y_1),\epsilon_1) \\
&= f_Y(t,\theta,\varphi,x,y,t-z \mid t',\theta_1,\varphi_1,x_1,y_1,t'-z_1) \\
&= f_{diff}(t,u,r|t',u_1,r_1)
\end{aligned}$$

$t-z=\epsilon$ $t'-z_1=\epsilon_1$ $u=(\theta,\varphi)$ $u_1=(\theta_1,\varphi_1)$ $r=(x,y,z)$ $r=(x_1,y_1,z_1)$

$f_{diff}(t,u,r \mid t',u_1,r_1)$ $\widetilde{f}_{diff}(t,u,r \mid t',u_1,r_1)$

7 The small angle approximation for total transport

[illegible]

If $T_{sa}(u') = A'$ and $T_{sa}(u'') = A''$ are vectors on the plane P corresponding to directions $u' \in S$ and $u'' \in S$ through small angle transformation T_{sa} (29) then the vector $A \in P$ denoting the sum of vectors A' and A'' on plane P ($A = A' + A''$) and the vector $T_{sa}(u) \in P$ denoting the vector corresponding to direction $u = u' + u'' \in S$ are equivalent in the sense of small angle approximation (i.e. the length of vector $A - T_{sa}(u) \in P$ is of the order higher than one relative to small parameter determining the length of the vector describing the particle's direction on plane P).

[illegible]

$$A(t) = T_{sa}(u(t)) = \sum_{i=0}^{N(t)} \left[A_{diff}(t_{i+1} - t_i) + A_i\right]$$

[illegible]

$$\frac{\partial f_{diff}(t, \theta_x, \theta_y | \theta'_x, \theta'_y)}{\partial t} = \frac{D}{2}\left(\frac{\partial^2}{\partial \theta_x^2} + \frac{\partial^2}{\partial \theta_y^2}\right) f_{diff}(t, \theta_x, \theta_y | \theta'_x, \theta'_y)$$

[illegible]

$$f_{diff}(0, \theta_x, \theta_y | \theta'_x, \theta'_y) = \delta(\theta_x - \theta'_x, \theta_y - \theta'_y)$$

A_i $(i = 1, ..., N(t))$ $\frac{1}{\sigma}\frac{d\sigma}{du}$

$p(A) = p(\theta_x, \theta_y)$ P

P

$$A(t) = \sum_{i=0}^{N(t)} A_{diff}(t_{i+1} - t_i) + \sum_{i=0}^{N(t)} A_i = A_{diff}(t) + \sum_{i=0}^{N(t)} A_i$$

$A(t)$ P $A_{diff}(t)$ $\sum_{i=1}^{N(t)} A$ P

$r(t)$ $u(t)$ $A(t)$ P dt u

$$udt = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)dt \in \mathbf{R}^3$$

t

$$r(t) = \int_0^t u(t')dt' \in \mathbf{R}^3, \quad r(0) = r_0$$

udt

$$udt = (A, 1)dt = (\theta\cos\varphi, \theta\sin\varphi, 1)dt = (\theta_x, \theta_y, 1)dt = (dx, dy, dz) \in \mathbf{R}^3$$

dt dz

t

$$r(t) = \int_0^t (A(t'), 1)dt' = (\int_0^t \theta_x(t')dt', \int_0^t \theta_y(t')dt', t) = (\rho(t), t) \in \mathbf{R}^3, \quad r(0) = r_0$$

$\rho(t) = \int_0^t A(t')dt'$ P'

$$\rho(t) = \int_0^t A(t')dt' = \int_0^t \left(A_{diff}(t') + \sum_{i=0}^{N(t')} A_i \right) dt' = \int_0^t A_{diff}(t')dt' + \int_0^t \left(\sum_{i=0}^{N(t')} A_i \right) dt' \in P'$$

$(u(t), r(t))$ $(A(t), \rho(t), t)$ $A(t)$ $\rho(t)$

$$\frac{\partial f_{pl}(t,A,\rho)}{\partial t} + A \cdot \nabla_\rho f_{pl}(t,A,\rho) = \frac{D}{2} \Delta f_{pl}(t,A,\rho) + \lambda \int_P \left[f_{pl}(t, A - \widetilde{A}, \rho) - f_{pl}(t,A,\rho) \right] p(\widetilde{A}) d\widetilde{A}$$

$f_{pl}(t,A,\rho)$ $f(t,u,r)$ z $z = t$ Δ P ∇_ρ P' $p(\widetilde{A})$ P $(A(t), \rho(t))$ $\left(A_{diff}(t), \int_0^t A_{diff}(t')dt' \right) \in P \times P'$ $\left(\sum_{i=0}^{N(t)} A_i, \int_0^t \left(\sum_{i=0}^{N(t')} A_i \right) dt' \right) \in P \times P'$

$$\begin{aligned}
(A(t), \rho(t)) &= \left(A_{diff}(t) + \sum_{i=0}^{N(t)} A_i, \int_0^t A_{diff}(t')dt' + \int_0^t \left(\sum_{i=0}^{N(t')} A_i \right) dt' \right) \\
&= \left(A_{diff}(t), \int_0^t A_{diff}(t')dt' \right) + \left(\sum_{i=0}^{N(t)} A_i, \int_0^t \left(\sum_{i=0}^{N(t')} A_i \right) dt' \right) \in P \times P'
\end{aligned}$$

$\left(A_{diff}(t), \int_0^t A_{diff}(t')dt' \right)$ $P \times P'$ $\int_0^t A_{diff}(t')dt'$ P' $\mathbf{R}^3$ $f_{diff}(t, A, \rho | t', A', \rho')$ $\left(\sum_{i=0}^{N(t)} A_i, \int_0^t \left(\sum_{i=0}^{N(t')} A_i \right) dt' \right)$ $P \times P'$ $\sum_{i=1}^{N(t)} A$ P $f_{coll}(t, A, \rho | t', A', \rho')$ $\left(\sum_{i=0}^{N(t)} A_i, \int_0^t \left(\sum_{i=0}^{N(t')} A_i \right) dt' \right)$ $P \times P'$

$$f_{coll}(t, A, \rho \mid t', A', \rho') = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} p^{(n)}(t, A, \rho \mid t', A', \rho')$$

$p^{(n)}(t, A, \rho \mid t', A', \rho')$ $\left(\sum_{i=0}^{N(t)} A_i, \int_0^t \left(\sum_{i=0}^{N(t')} A_i\right) dt'\right)$ $P \times P'$ (A', ρ') (A, ρ) (t', t) n (t', t)

$$p^{(n)}(t, A, \rho \mid t', A', \rho') = \frac{n}{t^n} \int_P [\int_{t'}^{t} p^{(n-1)}(\tau, A - A'', \rho - (t-\tau)A \mid t', A', \rho') \, p(A'')\tau^{n-1} d\tau] dA''$$

A $-A''$ ρ $-(t-\tau)A$ P P' $\left(A_{diff}(t), \int_0^t A_{diff}(t')dt'\right)$ $\left(\sum_{i=0}^{N(t)} A_i, \int_0^t \left(\sum_{i=0}^{N(t')} A_i\right) dt'\right)$ $f_{pl}(t, A, \rho | t', A', \rho')$ $(A(t), \rho(t))$ $f_{diff}(t, A, \rho | t', A', \rho')$ $f_{coll}(t, A, \rho | t', A', \rho')$

$$f_{pl}(t, A, \rho | t', A', \rho') = \int_{P \times P'} f_{diff}(t, A, \rho | t', A', \rho' - \rho'') f_{coll}(t, A - A'', \rho | t', A', \rho') dA'' d\rho''$$

$f_{pl}(t, A, \rho | t', A', \rho')$ $f(t, u, r | t', u', r')$ $f_{pl}(t, A, \rho | t', A', \rho')$

$$\begin{aligned} f_{pl}(t, A, \rho | t', A', \rho') &= f_{pl}(t, (\theta \cos\varphi, \theta \sin\varphi), (x, y) | t', (\theta' \cos\varphi', \theta' \sin\varphi'), (x', y')) \\ &= f(t, \theta, \varphi, x, y, z = t | t', \theta', \varphi', x', y', z' = t') \\ &= f(t, u, r | t', u', r') \end{aligned}$$

f_{pl} t (z, t) $z = t$ $z = t$ $t^2 > z^2$ $\rho^2 =$

$x^2 + y^2 > 0$

f_{pl} (x, y, t) (z, x, y, t)

$t^2 = x^2 + y^2 + z^2$ (z, t) $z = t$

$$
\begin{aligned}
f_{pl}(t, A, \rho | t', A', \rho') &= f_{pl}(t, (\theta\cos\varphi, \theta\sin\varphi), (x, y) | t', (\theta'\cos\varphi', \theta'\sin\varphi'), (x', y')) \\
&= f(t, \theta, \varphi, x, y, z = \sqrt{t^2 - x^2 - y^2} | t', \theta', \varphi', x', y', z' = \sqrt{(t')^2 - (x')^2 - (y')^2}) \\
&= f(t, u, r | t', u', r')
\end{aligned}
$$

References

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325

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[illegible] **98**

[illegible] **52** .

[illegible] **22**

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