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ANGULAR DEPENDENT COARSE-MESH REBALANCE METHOD FOR ACCELERATION OF THE DISCRETE ORDINATES NEUTRON TRANSPORT CALCULATIONS

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ABSTRACT

A new coarse-mesh rebalance method is developed and tested to accelerate one-dimensional discrete ordinates neutron transport calculations. The method is based on the use of rebalance factors that are angular dependent and defined only on the coarse-mesh boundaries. Unlike the original Coarse-Mesh Rebalance (CMR) method that is only conditionally stable, Fourier analysis and numerical results show that this Angular Dependent Coarse-Mesh Rebalance (ADCMR) method is unconditionally stable for any optical thickness, scattering ratio, and coarseness and that the acceleration is very effective in most cases.

Key Words: acceleration, CMR, angular dependent rebalance, Fourier analysis, unconditional stability

1. INTRODUCTION

Since Kopp[1] and Lebedev[2] introduced the synthetic concept to accelerate source iteration (SI) of the transport calculations in early sixties, various acceleration methods were proposed such as conventional rebalance methods, Diffusion Synthetic Acceleration (DSA)[3,4,5], Transport Synthetic Acceleration (TSA)[6], Projected Discrete Ordinates (PDO)[7,8,9], Weighted Alpha (WA)[10,11,12], and so on.[13] Coarse-Mesh Rebalance (CMR) method[14,15] that was once the most popular acceleration method has existed in the literature since the mid-1960s and was implemented in many commercial neutron transport codes. CMR method is versatile in that it can be applied to a wide range of geometries with any S_N differencing scheme. However, CMR is unstable with scattering ratio c close to unity or with optically thick cells. Cefus and Larsen[15] showed this analytically by using Fourier analysis. Thus the unconditionally stable DSA method developed in late 1970s and early 1980s replaced CMR in many codes. Although DSA shows better behavior in spectral radius, DSA still has a problem. For unconditional stability, DSA needs consistency in spatial discretizations of high and low order equations, that is not easy to accomplish in multi-dimensional and non-diamond differenced schemes.

Due to expansion of computer capacity, people wish to solve three-dimensional whole-core heterogeneous problems for accuracy. Even though computer technology is advancing, it needs many iterations and long computing time. Because CMR is based on neutron balance over the coarse-mesh only, it will be possible to apply it to other than S_N methods such as Method Of Characterics (MOC) and treat irregular or unstructured meshes overlayed by simple coarse meshes. These aspects are very attractive but the original CMR does not show unconditional stability. For these reasons, we will discuss an unconditionally stable coarse-mesh rebalance method that is accomplished by introducing rebalance factors that are angular dependent and defined only on coarse-mesh boundaries. The angular dependent rebalance factors were already introduced in fine-mesh cases by Hong and Cho.[16,17] This angular dependent rebalance (ADR) method is unconditionally stable and shows good convergence. A related but distinct method called additive angular dependent rebalance (AADR) method was proposed in fine mesh cases by Park and Cho.[18,19] In AADR, rebalance factors are introduced before spatial discretization, while in ADR, reblance facotrs are introduced after discretization. Also in fine meshes, Adams[20] combined an angular shape function concept with linear boundary projection acceleration[21] in a one-dimensional nonlinear acceleration method.

As a starting point we will describe Angular Dependent Coarse-Mesh Rebalance method, shortly ADCMR, for one-dimensional S_N equation and demonstrate unconditional stability by Fourier analysis and numerical results.

2. FORMULATION

Let us consider slab geometry like in Fig. 1. The whole problem consists of N coarse meshes, each coarse mesh containing p fine-meshes. The l-th source iteration for the S_N transport calculation on a nonuniform mesh is described by

$$\frac{\mu_m}{h_i}(\psi_{m,i+1/2}^{l+1/2}-\psi_{m,i-1/2}^{l+1/2})+\sigma_{t,i}\psi_{m,i}^{l+1/2}=\frac{1}{2}(\sigma_{s,i}\phi_i^l+Q_i), \quad (1)$$

$$\phi_i^{l+1/2}=\sum_m w_m\psi_{m,i}^{l+1/2}, \quad (2)$$

$$\sum_m w_m=2, \quad (3)$$

$$\phi_i^{l+1}=\phi_i^{l+1/2}. \quad (4)$$

Then mesh-centered angular fluxes can be expressed as

$$\psi_{m,i}^{l+1/2}=\frac{1+\alpha_{m.i}}{2}\psi_{m,i+1/2}^{l+1/2}+\frac{1-\alpha_{m.i}}{2}\psi_{m,i-1/2}^{l+1/2} \quad (5)$$

where

$$\alpha_{m,i}=0 \qquad \text{for diamond difference (DD) scheme,} \quad (6a)$$

$$=\frac{\mu_m}{|\mu_m|} \qquad \text{for step difference (SD) scheme,} \quad (6b)$$

$$=\frac{e^{\varepsilon_{m,i}}+e^{-\varepsilon_{m,i}}}{e^{\varepsilon_{m,i}}-e^{-\varepsilon_{m,i}}}-\frac{1}{\varepsilon_{m,i}},\quad \varepsilon_{m,i}=\frac{\sigma_{t,i}h_i}{2\mu_m} \qquad \text{for step characteristic (SC) scheme.} \quad (6c)$$

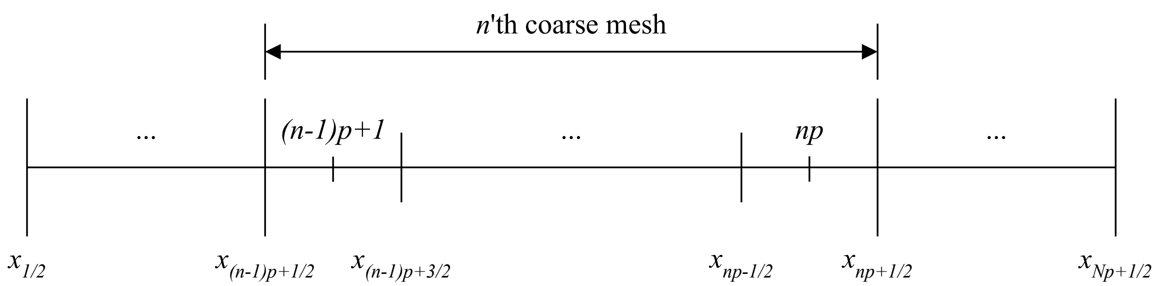

Fig. 1. Coarse-mesh divisions of slab geometry.

By using a chosen spatial discretization scheme, the outgoing and mesh-centered angular fluxes can be expressed as follows using incoming angular flux in the *i*-th fine mesh:

$$\begin{aligned} \psi_{m,i+1/2}^{l+1/2} &= A_{m,i}^{+}\psi_{m,i-1/2}^{l+1/2} + B_{m,i}^{+}(\sigma_{s,i}\phi_i^l + Q_i), \\ \psi_{m,i-1/2}^{l+1/2} &= A_{m,i}^{-}\psi_{m,i+1/2}^{l+1/2} + B_{m,i}^{-}(\sigma_{s,i}\phi_i^l + Q_i), \end{aligned} \tag{7a}$$

$$\begin{aligned} \psi_{m,i}^{l+1/2} &= C_{m,i}^{+}\psi_{m,i-1/2}^{l+1/2} + D_{m,i}^{+}(\sigma_{s,i}\phi_i^l + Q_i), \\ \psi_{m,i}^{l+1/2} &= C_{m,i}^{-}\psi_{m,i+1/2}^{l+1/2} + D_{m,i}^{-}(\sigma_{s,i}\phi_i^l + Q_i). \end{aligned} \tag{7b}$$

The constants *A*, *B*, *C*, and *D* are given by

$$A_{m,i}^{+} = \frac{2\mu_m - \sigma_{t,i}h_i(1-\alpha_{m,i})}{2\mu_m + \sigma_{t,i}h_i(1+\alpha_{m,i})}, \qquad A_{m,i}^{-} = \frac{2\mu_m + \sigma_{t,i}h_i(1+\alpha_{m,i})}{2\mu_m - \sigma_{t,i}h_i(1-\alpha_{m,i})}, \tag{8a}$$

$$B_{m,i}^{+} = \frac{h_i}{2\mu_m + \sigma_{t,i}h_i(1+\alpha_{m,i})}, \qquad B_{m,i}^{-} = \frac{-h_i}{2\mu_m - \sigma_{t,i}h_i(1-\alpha_{m,i})}, \tag{8b}$$

$$C_{m,i}^{+} = \frac{2\mu_m}{2\mu_m + \sigma_{t,i}h_i(1+\alpha_{m,i})}, \qquad C_{m,i}^{-} = \frac{2\mu_m}{2\mu_m - \sigma_{t,i}h_i(1-\alpha_{m,i})}, \tag{8c}$$

$$D_{m,i}^{+} = \frac{h_i(1+\alpha_{m,i})/2}{2\mu_m + \sigma_{t,i}h_i(1+\alpha_{m,i})}, \quad \text{and} \quad D_{m,i}^{-} = \frac{-h_i(1-\alpha_{m,i})/2}{2\mu_m - \sigma_{t,i}h_i(1-\alpha_{m,i})}. \tag{8d}$$

After eliminating all interior angular fluxes successively, the outgoing angular fluxes of the *n*-th coarse mesh and all mesh-centered angular fluxes are expressed as

$$\begin{aligned} \psi_{m,np+1/2}^{l+1/2} &= \prod_{i=(n-1)p+1}^{np} A_{m,i}^{+}\psi_{m,(n-1)p+1/2}^{l+1/2} + \sum_{i=(n-1)p+1}^{np}\prod_{k=i+1}^{np} A_{m,k}^{+}B_{m,i}^{+}(\sigma_{s,i}\phi_i^l + Q_i), \\ \psi_{m,(n-1)p+1/2}^{l+1/2} &= \prod_{i=(n-1)p+1}^{np} A_{m,i}^{-}\psi_{m,np+1/2}^{l+1/2} + \sum_{i=(n-1)p+1}^{np}\prod_{k=(n-1)p+1}^{i-1} A_{m,k}^{-}B_{m,i}^{-}(\sigma_{s,i}\phi_i^l + Q_i), \end{aligned} \tag{9a}$$

$$
\begin{aligned}
\psi_{m,i}^{l+1/2} &= C_{m,i}^{+} \prod_{k=(n-1)p+1}^{i-1} A_{m,k}^{+} \psi_{m,(n-1)p+1/2}^{l+1/2} \\
&\quad + C_{m,i}^{+} \sum_{k=(n-1)p+1}^{i-1} \prod_{u=k+1}^{i-1} A_{m,u}^{+} B_{m,k}^{+} (\sigma_{s,k}\phi_k^l + Q_k) + D_{m,i}^{+} (\sigma_{s,i}\phi_i^l + Q_i), \\
\psi_{m,i}^{l+1/2} &= C_{m,i}^{-} \prod_{k=i+1}^{np} A_{m,k}^{-} \psi_{m,np+1/2}^{l+1/2} \\
&\quad + C_{m,i}^{-} \sum_{k=i+1}^{np} \prod_{u=i+1}^{k-1} A_{m,u}^{-} B_{m,k}^{-} (\sigma_{s,k}\phi_k^l + Q_k) + D_{m,i}^{-} (\sigma_{s,i}\phi_i^l + Q_i),
\end{aligned}
\tag{9b}
$$

for $i = (n-1)p+1, (n-1)p+2, \cdots, np$.

The nonlinear rebalance factors f 's, which are defined on coarse-mesh boundaries only, are simply expressed as follows:

$$
f_{n+1/2}^{+} = \frac{\psi_{m,np+1/2}^{l+1}}{\psi_{m,np+1/2}^{l+1/2}}, \ \mu_m > 0, \tag{10a}
$$

$$
f_{n+1/2}^{-} = \frac{\psi_{m,np+1/2}^{l+1}}{\psi_{m,np+1/2}^{l+1/2}}, \ \mu_m < 0. \tag{10b}
$$

The factor in Eqs. (10a) and (10b) are defined as ratios of the new iterate flux to the previous iterate, similarly to the CMR method. However, it is angular dependent and defined only on the mesh boundaries, while the factor in CMR is an angle-independent constant defined over the mesh. The DP_0-like (or S_2-like) representation in Eqs. (10a) and (10b) is a special case of the more general DP_N-like (or S_N-like) formalism.[16,17]

Now we change all the iteration indices to $l+1$ and introduce rebalance factors given in Eqs. (10a) and (10b). Then by multiplying Eq. (9a) by μ_m and summing over half angle, we find equations for the rebalance factors as

$$
\begin{aligned}
\left(\sum_{\mu_m>0} \mu_m w_m \psi_{m,np+1/2}^{l+1/2} \right) f_{n+1/2}^{+} &= \left(\sum_{\mu_m>0} \mu_m w_m \prod_{i=(n-1)p+1}^{np} A_{m,i}^{+} \psi_{m,(n-1)p+1/2}^{l+1/2} \right) f_{n-1/2}^{+} \\
&\quad + \sum_{i=(n-1)p+1}^{np} \left(\sum_{\mu_m>0} \mu_m w_m \prod_{k=i+1}^{np} A_{m,k}^{+} B_{m,i}^{+} \right) (\sigma_{s,i}\phi_i^{l+1} + Q_i),
\end{aligned}
\tag{11a}
$$

$$
\begin{aligned}
\left(\sum_{\mu_m<0} \mu_m w_m \psi_{m,(n-1)p+1/2}^{l+1/2} \right) f_{n-1/2}^{-} &= \left(\sum_{\mu_m<0} \mu_m w_m \prod_{i=(n-1)p+1}^{np} A_{m,i}^{-} \psi_{m,np+1/2}^{l+1/2} \right) f_{n+1/2}^{-} \\
&\quad + \sum_{i=(n-1)p+1}^{np} \left(\sum_{\mu_m<0} \mu_m w_m \prod_{k=(n-1)p+1}^{i-1} A_{m,k}^{-} B_{m,i}^{-} \right) (\sigma_{s,i}\phi_i^{l+1} + Q_i),
\end{aligned}
\tag{11b}
$$

and by summing Eq. (9b) over half angle, we find the mesh-centered scalar fluxes as follows:

$$\begin{aligned}\phi_i^{l+1,+} &= \left(\sum_{\mu_m>0} w_m C_{m,i}^{+} \prod_{k=(n-1)p+1}^{i-1} A_{m,k}^{+} \psi_{m,(n-1)p+1/2}^{l+1/2} \right) f_{n-1/2}^{+} \\ &\quad + \sum_{k=(n-1)p+1}^{i-1} \sum_{\mu_m>0} w_m C_{m,i}^{+} \prod_{u=k+1}^{i-1} A_{m,u}^{+} B_{m,k}^{+} (\sigma_{s,k}\phi_k^{l+1} + Q_k) + \sum_{\mu_m>0} w_m D_{m,i}^{+} (\sigma_{s,i}\phi_i^{l+1} + Q_i),\end{aligned} \tag{12a}$$

$$\begin{aligned}\phi_i^{l+1,-} &= \left(\sum_{\mu_m<0} w_m C_{m,i}^{-} \prod_{k=i+1}^{np} A_{m,k}^{-} \psi_{m,np+1/2}^{l+1/2} \right) f_{n+1/2}^{-} \\ &\quad + \sum_{k=i+1}^{np} \sum_{\mu_m<0} w_m C_{m,i}^{-} \prod_{u=i+1}^{k-1} A_{m,u}^{-} B_{m,k}^{-} (\sigma_{s,k}\phi_k^{l+1} + Q_k) + \sum_{\mu_m<0} w_m D_{m,i}^{-} (\sigma_{s,i}\phi_i^{l+1} + Q_i),\end{aligned} \tag{12b}$$

$$\phi_i^{l+1} = \phi_i^{l+1,+} + \phi_i^{l+1,-}, \tag{12c}$$

for $i = (n-1)p+1, (n-1)p+2, \cdots, np$.

The overall procedure of our ADCMR method can be described as follows. First, the high-order transport equations are solved by transport sweep and the coefficients in Eqs. (11a), (11b), (12a), and (12b) are calculated. Second, the low-order equations (11a, 11b, 12a, 12b) are solved by iteration. Third, the final converged solutions of the low-order equations are used in the high-order equations. The procedure is repeated until the scalar flux converges in each mesh. This procedure is shown in Fig. 2.

3. FOURIER ANALYSIS

Thus far the acceleration equations of the ADCMR method, which is a nonlinear scheme, have been derived in slab geometry. However, it is yet to be proved theoretically that the acceleration equations derived actually accelerate the transport equation. The most popular technique that analyzes iterative schemes is Fourier stability analysis that can apply only to linear methods. But Cefus and Larsen successfully applied this technique to the analysis of CMR and PDO iterative schemes through linearization. In this section we analyze theoretically the ADCMR method by the Cefus and Larsen's approach. To evade the complexity of ADCMR, a special class of infinite homogeneous medium problems with a flat source is considered. Therefore, the medium has the simple solution given by $\phi = Q/\sigma_a$. Then, the ADCMR equations are linearized around this solution.

Let

$$\begin{aligned}\psi_{m,i+1/2} &= \frac{Q}{\sigma_a}\left(\frac{1}{2} + \varepsilon \xi_{m,i+1/2}\right), \\ \phi_i^{\pm} &= \frac{Q}{\sigma_a}\left(\frac{1}{2} + \varepsilon \varsigma_i^{\pm}\right), \\ f^{\pm} &= 1 + \varepsilon F^{\pm}.\end{aligned} \tag{13}$$

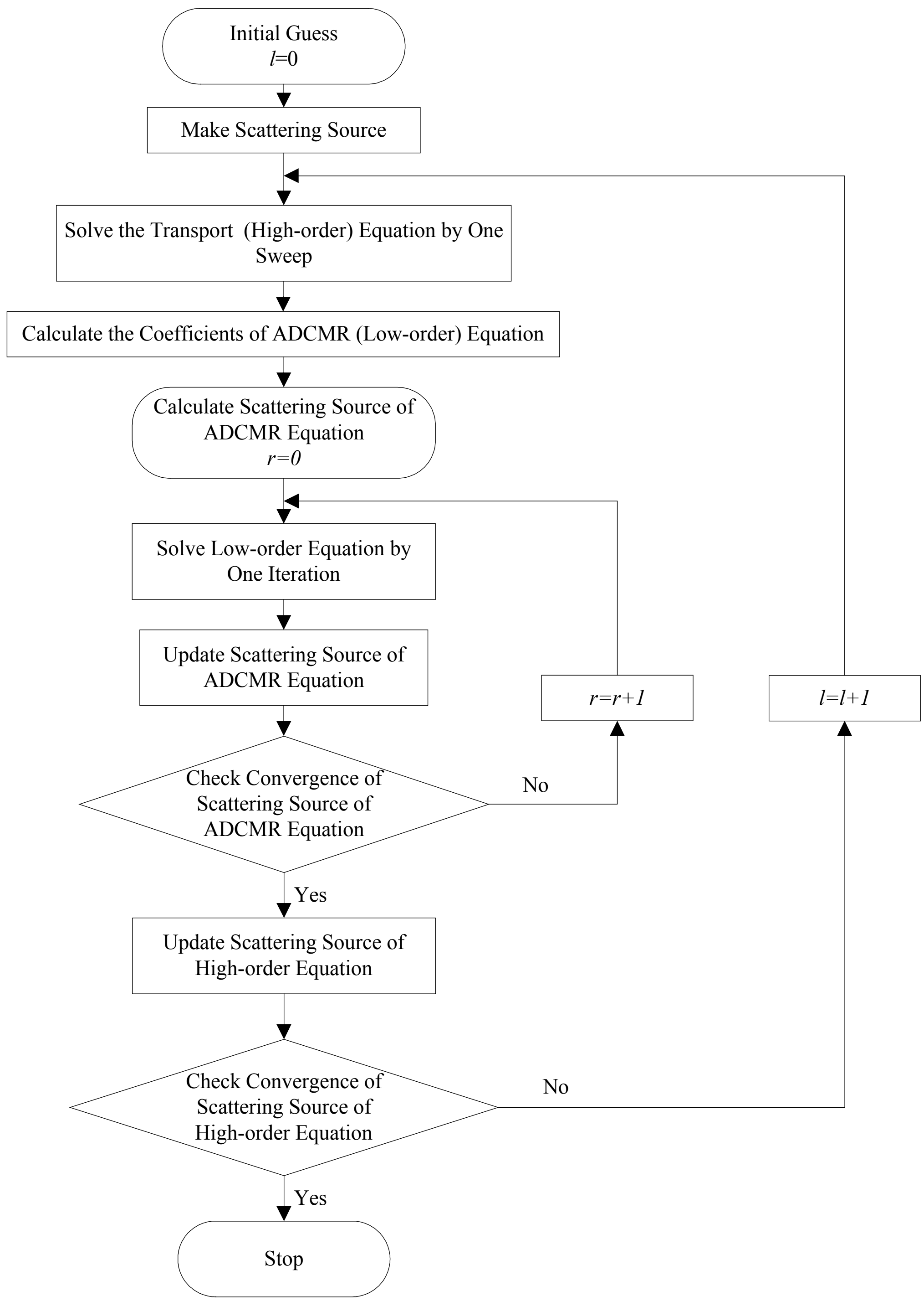

Fig. 2. The overall procedure of the ADCMR method.

Inserting Eq. (13) into the balance equation of the coarse-mesh (10a) and low-order equations (11a, 11b, 12a, and 12b) and dropping $O(\varepsilon^2)$ terms, we find a system of linear equations as follows:

$$\begin{aligned}
\xi_{m,p+1/2}^{l+1/2} &= (A_m^+)^p \xi_{m,1/2}^{l+1/2} + \sigma_s \sum_{i=1}^{p} (A_m^+)^{p-i} B_m^+ (\varsigma_i^{l+1,+} + \varsigma_i^{l+1,-}), \\
\xi_{m,1/2}^{l+1/2} &= (A_m^-)^p \xi_{m,p+1/2}^{l+1/2} + \sigma_s \sum_{i=1}^{p} (A_m^-)^{i-1} B_m^- (\varsigma_i^{l+1,+} + \varsigma_i^{l+1,-}),
\end{aligned} \tag{14a}$$

$$\begin{aligned}
&\sum_{\mu_m>0} \mu_m w_m (\xi_{m,p+1/2}^{l+1/2} + \frac{1}{2} F_{3/2}^+) \\
&\quad = \sum_{\mu_m>0} \mu_m w_m (A_m^+)^p (\xi_{m,1/2}^{l+1/2} + \frac{1}{2} F_{1/2}^+) + \sigma_s \sum_{i=1}^{p} \left(\sum_{\mu_m>0} \mu_m w_m (A_m^+)^{p-i} B_m^+ \right) (\varsigma_i^{l+1,+} + \varsigma_i^{l+1,-}), \\
&\sum_{\mu_m<0} \mu_m w_m (\xi_{m,1/2}^{l+1/2} + \frac{1}{2} F_{1/2}^-) \\
&\quad = \sum_{\mu_m<0} \mu_m w_m (A_m^-)^p (\xi_{m,p+1/2}^{l+1/2} + \frac{1}{2} F_{3/2}^-) + \sigma_s \sum_{i=1}^{p} \left(\sum_{\mu_m<0} \mu_m w_m (A_m^-)^{i-1} B_m^- \right) (\varsigma_i^{l+1,+} + \varsigma_i^{l+1,-}),
\end{aligned} \tag{14b}$$

$$\begin{aligned}
\varsigma_i^{l+1,+} &= \sum_{\mu_m>0} w_m C_m^+ (A_m^+)^{i-1} (\xi_{m,1/2}^{l+1/2} + \frac{1}{2} F_{1/2}^+) + \sigma_s \sum_{k=1}^{i-1} \left(\sum_{\mu_m>0} w_m C_m^+ (A_m^+)^{i-1-k} B_m^+ \right) (\varsigma_k^{l+1,+} + \varsigma_k^{l+1,-}) \\
&\quad + \sigma_s \left(\sum_{\mu_m>0} w_m D_m^+ \right) (\varsigma_i^{l+1,+} + \varsigma_i^{l+1,-}), \\
\varsigma_i^{l+1,-} &= \sum_{\mu_m<0} w_m C_m^- (A_m^-)^{p-i} (\xi_{m,p+1/2}^{l+1/2} + \frac{1}{2} F_{3/2}^-) + \sigma_s \sum_{k=i+1}^{p} \left(\sum_{\mu_m<0} w_m C_m^- (A_m^-)^{k-(i+1)} B_m^- \right) (\varsigma_k^{l+1,+} + \varsigma_k^{l+1,-}) \\
&\quad + \sigma_s \left(\sum_{\mu_m<0} w_m D_m^- \right) (\varsigma_i^{l+1,+} + \varsigma_i^{l+1,-}).
\end{aligned} \tag{14c}$$

In the equations above, the coarse-mesh index *n* and fine-mesh index *i* in constants A, B, C, and D disappear because the medium under consideration is infinite and homogeneous.

Next, the following Fourier ansatz are chosen:

$$\begin{aligned}
\xi_{m,1/2}^{l+1/2} &= b_m e^{j\lambda x_{1/2}}, \qquad & \xi_{m,p+1/2}^{l+1/2} &= b_m e^{j\lambda x_{p+1/2}}, \\
\varsigma_i^{l+1/2,\pm} &= a_i^{\pm} e^{j\lambda x_i}, \qquad & \varsigma_i^{l+1,\pm} &= \omega a_i^{\pm} e^{j\lambda x_i}, \\
F_{1/2}^{\pm} &= G^{\pm} e^{j\lambda x_{1/2}}, \qquad & F_{3/2}^{\pm} &= G^{\pm} e^{j\lambda x_{p+1/2}}.
\end{aligned} \tag{15}$$

Substituting the Fourier ansatz Eq. (15) into Eq. (14) gives following equations:

$$\begin{aligned}\omega a_i^+ &= \sigma_s \sum_{k=1}^{p}\left[\alpha_{ik}^+ + (\omega-1)\beta_{ik}^+\right](a_k^+ + a_k^-) + \sigma_s\omega\sum_{k=1}^{i-1}\gamma_{ik}^+(a_k^+ + a_k^-) + \sigma_s\omega\delta^+,\\ \omega a_i^- &= \sigma_s \sum_{k=1}^{p}\left[\alpha_{ik}^- + (\omega-1)\beta_{ik}^-\right](a_k^+ + a_k^-) + \sigma_s\omega\sum_{k=1}^{i-1}\gamma_{ik}^-(a_k^+ + a_k^-) + \sigma_s\omega\delta^-,\end{aligned} \tag{16}$$

where

$$\begin{aligned}\alpha_{ik}^+ &= \exp[2(k-i)\tau j]\sum_{\mu_m>0}\frac{w_m(A_m^+)^{p+(i-1)-k}B_m^+C_m^+}{\exp[2p\tau j]-(A_m^+)^p},\\ \alpha_{ik}^- &= \exp[2(k-i)\tau j]\sum_{\mu_m<0}\frac{w_m(A_m^-)^{p+k-(i+1)}B_m^-C_m^-}{\exp[-2p\tau j]-(A_m^-)^p},\end{aligned} \tag{17a}$$

$$\begin{aligned}\beta_{ik}^+ &= \exp[2(k-i)\tau j]\frac{\sum_{\mu_m>0}w_m(A_m^+)^{i-1}C_m^+\sum_{\mu_m>0}\mu_m w_m(A_m^+)^{p-k}B_m^+}{\sum_{\mu_m>0}\mu_m w_m\left(\exp[2p\tau j]-(A_m^+)^p\right)},\\ \beta_{ik}^- &= \exp[2(k-i)\tau j]\frac{\sum_{\mu_m<0}w_m(A_m^-)^{p-i}C_m^-\sum_{\mu_m<0}\mu_m w_m(A_m^-)^{k-1}B_m^-}{\sum_{\mu_m<0}\mu_m w_m\left(\exp[-2p\tau j]-(A_m^-)^p\right)},\end{aligned} \tag{17b}$$

$$\begin{aligned}\gamma_{ik}^+ &= \exp[2(k-i)\tau j]\sum_{\mu_m>0}w_m(A_m^+)^{i-1-k}B_m^+,\\ \gamma_{ik}^- &= \exp[2(k-i)\tau j]\sum_{\mu_m<0}w_m(A_m^-)^{k-(i+1)}B_m^-,\end{aligned} \tag{17c}$$

$$\begin{aligned}\delta^+ &= \sum_{\mu_m>0}w_m D_m^+,\\ \delta^- &= \sum_{\mu_m<0}w_m D_m^-.\end{aligned} \tag{17d}$$

Eq. (16) can be rewritten in matrix form as follows:

$$\begin{bmatrix}\sigma_s\mathbf{A}_1^+ + \omega(\sigma_s\mathbf{A}_2^+ - \mathbf{I}) & \sigma_s\mathbf{A}_1^+ + \omega\sigma_s\mathbf{A}_2^+\\ \sigma_s\mathbf{A}_1^- + \omega\sigma_s\mathbf{A}_2^- & \sigma_s\mathbf{A}_1^- + \omega(\sigma_s\mathbf{A}_2^- - \mathbf{I})\end{bmatrix}\begin{bmatrix}\mathbf{a}^+\\ \mathbf{a}^-\end{bmatrix} = \mathbf{0}, \tag{18}$$

where $\mathbf{I}$ is $p\times p$ identity matrix and

$$\mathbf{a}^+ = \begin{bmatrix}a_1^+ & \cdots & a_p^+\end{bmatrix}^T, \qquad \mathbf{a}^- = \begin{bmatrix}a_1^- & \cdots & a_p^-\end{bmatrix}^T, \tag{19a}$$

$$\mathbf{A}_1^+ = \begin{bmatrix}\alpha_{11}^+ - \beta_{11}^+ & \dots & \alpha_{1p}^+ - \beta_{1p}^+\\ \vdots & \ddots & \vdots\\ \alpha_{p1}^+ - \beta_{p1}^+ & \cdots & \alpha_{pp}^+ - \beta_{pp}^+\end{bmatrix}, \qquad \mathbf{A}_1^- = \begin{bmatrix}\alpha_{11}^- - \beta_{11}^- & \dots & \alpha_{1p}^- - \beta_{1p}^-\\ \vdots & \ddots & \vdots\\ \alpha_{p1}^- - \beta_{p1}^- & \cdots & \alpha_{pp}^- - \beta_{pp}^-\end{bmatrix}, \tag{19b}$$

$$\mathbf{A}_2^+ = \begin{bmatrix} \beta_{11}^+ + \delta^+ & \beta_{12}^+ & & \cdots & \beta_{1p}^+ \\ \beta_{21}^+ + \gamma_{21}^+ & \beta_{22}^+ + \delta^+ & \beta_{23}^+ & & \vdots \\ \vdots & & & \ddots & \beta_{p-1p}^+ \\ \beta_{p1}^+ + \gamma_{p1}^+ & \beta_{p2}^+ + \gamma_{p2}^+ & & \beta_{pp-1}^+ + \gamma_{pp-1}^+ & \beta_{pp}^+ + \delta^+ \end{bmatrix},$$

$$\mathbf{A}_2^- = \begin{bmatrix} \beta_{11}^- + \delta^- & \beta_{12}^- + \gamma_{12}^- & & \cdots & \beta_{1p}^- + \gamma_{1p}^- \\ \beta_{21}^- & \beta_{22}^- + \delta^- & \beta_{23}^- + \gamma_{23}^- & & \vdots \\ \vdots & & & \ddots & \beta_{p-1p}^- + \gamma_{p-1p}^- \\ \beta_{p1}^- & \beta_{p2}^- & & \beta_{pp-1}^- & \beta_{pp}^- + \delta^- \end{bmatrix}. \tag{19c}$$

Therefore, the eigenvalues ω's of the iteration operator of ADCMR are obtained by equating the determinant of Eq. (18) to zero. The spectral radius ρ is then the largest absolute value of ω's.

$$\rho = \sup_\tau |\omega(\tau)|. \tag{20}$$

For the DD discretization scheme, we can find the following properties:

$$\mathbf{A}_1^- = \overline{\mathbf{A}_1^+}^T, \qquad \mathbf{A}_2^- = \overline{\mathbf{A}_2^+}^T, \tag{21}$$

where $\overline{\mathbf{X}}^T$ denotes transpose matrix of $\mathbf{X}$ with complex elements conjugated.

4. NUMERICAL AND FOURIER ANALYSIS RESULTS

4.1. Benchmark Problems

Several benchmark problems are tested. The DD scheme is used for all problems and the same error criterion is used for high- and low-order equations. From the benchmark results we find that ADCMR is unconditionally stable in not only homogeneous but also heterogeneous medium with optically thick mesh size. In all problems the same error criteria are used for low order equations and high order equations.

4.1.1. Benchmark I (Reed's Problem)

- Infinite slab with homogeneous medium
- Constant mesh size
- Vacuum boundary conditions on both sides
- S_6
- Error criteria ($\max_i \left|1-\phi_i^{l+1}/\phi_i^l\right|$) : 10^{-4}

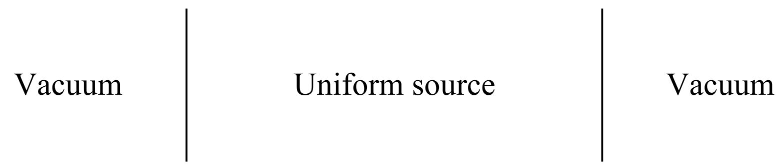

Fig. 3. Benchmark problem I.

Table I. Material data of benchmark problem I

Problem	*c*	$\sigma_t\,(cm^{-1})$	*h* (cm^{-1})	*Q* $(cm^{-3}sec^{-1})$	Total size (*cm*)
M1	1.0	1.0	1.0	1.0	30.0
M2	1.0	2.0	1.0	1.0	30.0
M3	1.0	1.0	1.0	1.0	60.0
M4	1.0	1.0	2.0	1.0	60.0

Table II. Number of iterations of benchmark problem I

	SI[a]	ADR (*p=1*)	ADCMR				
			p=2	*p=5*	*p=10*	*p=15*	*p=30*
M1	1072	4	5	4	4	3	2
M2	2641	5	5	4	4	4	2
M3	2644	5	5	4	4	4	4
M4	2642	5	5	4	4	4	2

[a] : source iteration

4.1.2. Benchmark II (Khalil's Problem)

- Uniform isotropic scattering medium with various cross sections with $c=0.98$
- Constant source in the left half of the slab
- $h=1cm$
- S_{16}
- Error criteria : 10^{-4}

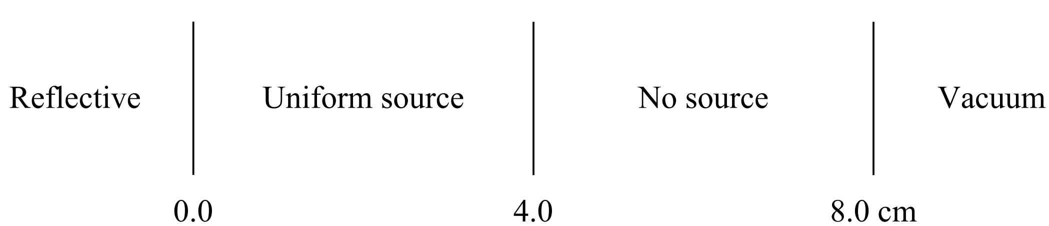

Fig. 4. Benchmark problem II.

Table III. Number of iterations of benchmark problem II

$\sigma_t\,(cm^{-1})$		1.0	2.0	4.0	6.0	10.0	20.0
SI		197	272	358	504	448	394
ADR (*p=1*)		5	5	5	6	7	5
ADCMR	*p=2*	5	5	5	6	7	5
	p=4	4	4	4	6	6	5

4.1.2. Benchmark III (Modified Adams and Martin's Problem)

- Highly heterogeneous problem
- 10 meshes per material
- S_{16}
- Error criteria : 10^{-6}

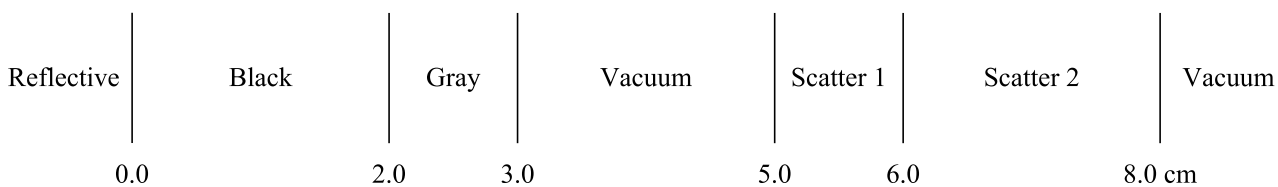

Fig. 5. Benchmark problem III.

Table IV. Material data of benchmark problem III

	Black	Gray	Scatter 1	Scatter 2
$Q\,(cm^{-3}sec^{-1})$	50.0	0.0	1.0	0.0
$\sigma_t\,(cm^{-1})$	50.0	5.0	2.0	1.0
c	0.0	0.0	1.0	1.0

Table V. Number of iterations of benchmark problem III

SI	ADR (*p=1*)	ADCMR		
		P=2	*p=5*	*p=10*
112	6	6	6	6

4.2. Results of Fourier Analysis

In addition to the numerical results, we can confirm that ADCMR is unconditionally stable with several discretization schemes by the results of Fourier analysis. Fig. 6 shows the spectral radii for several scattering ratios when coarseness *p=2* with DD scheme. The smaller scattering ratio gives always better results. Fig. 7 shows that spectral radius decreases as coarseness (p) increases. Fig. 8 shows that ADCMR is stable for DD, SD, and SC discretization schemes. In comparison to the conventional CMR, ADCMR is not only more stable but also unconditionally stable and effective (Fig. 7, Fig. 9, and Fig. 10).

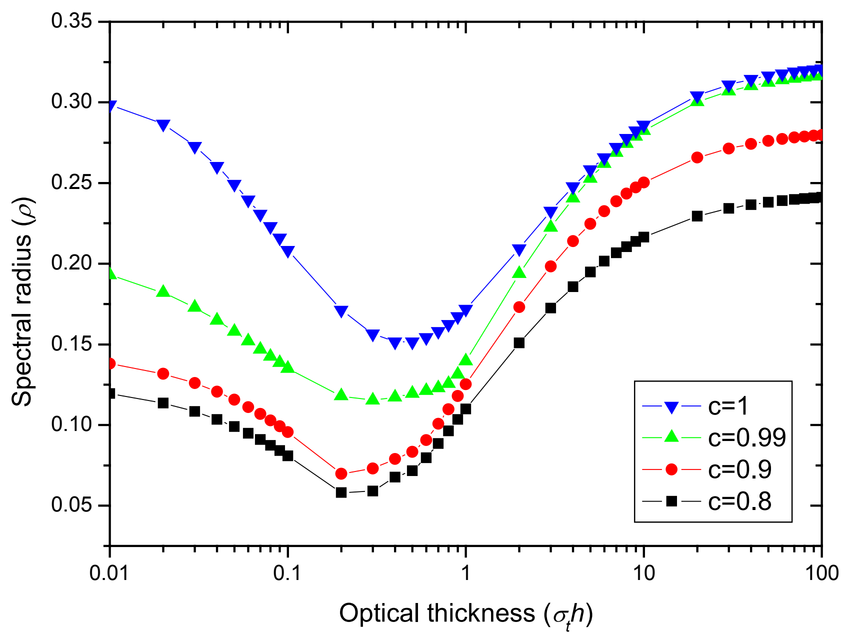

Figure 6. Spectral radius of ADCMR with DD when *p=2* for various scattering ratios.

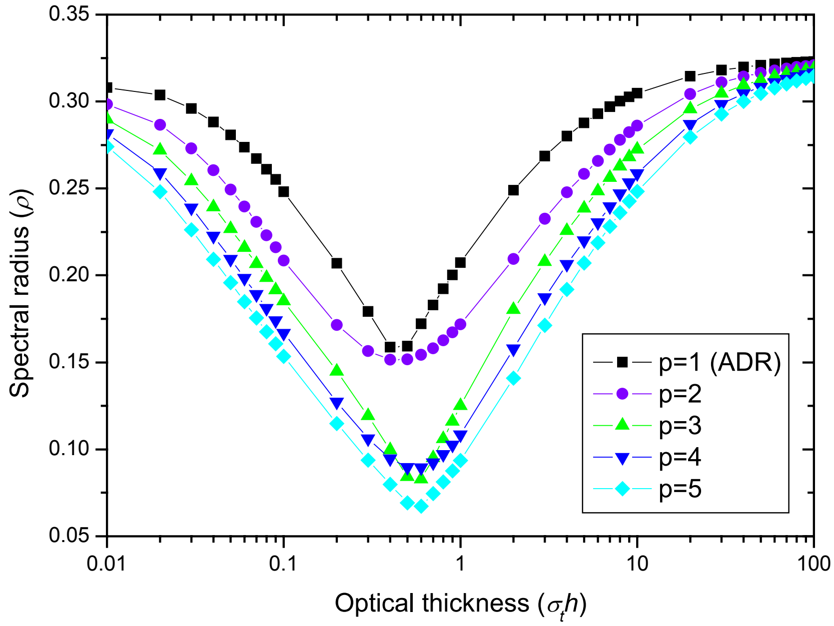

Figure 7. Spectral radius of ADCMR with DD for various *p* with *c=1.*

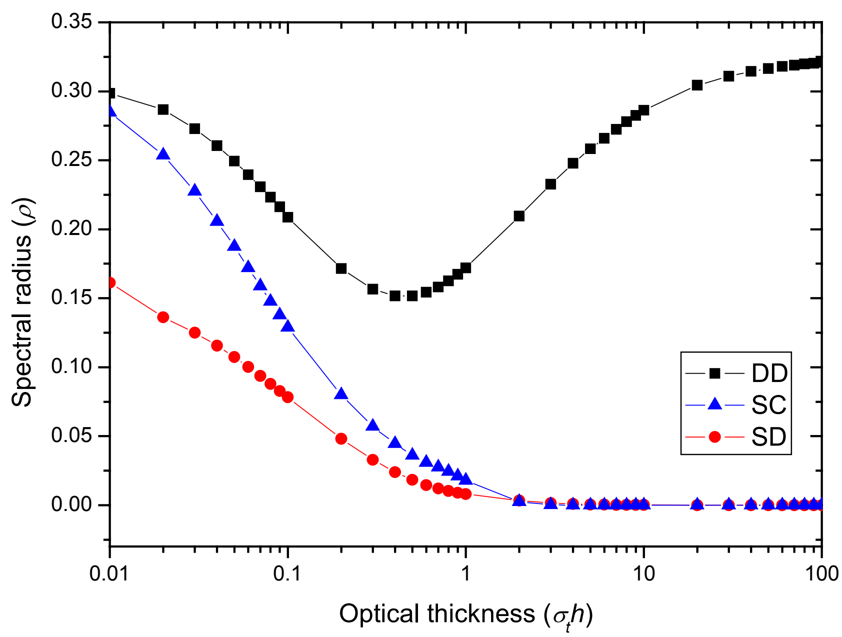

Figure 8. Spectral radius of ADCMR for DD, SC, and SD discretization schemes with *p*=2 and *c*=1.

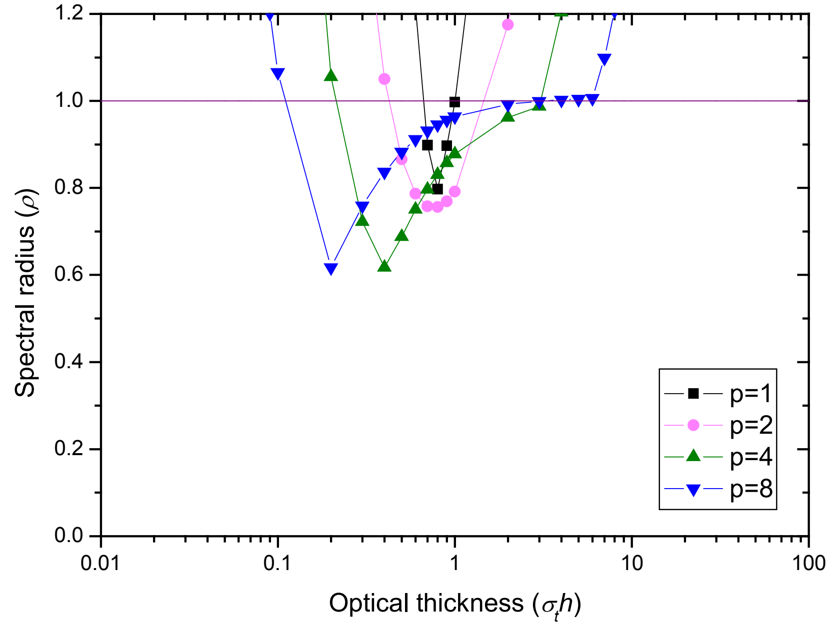

Figure 9. Spectral radius of CMR with DD for various *p*'s with *c=1*.

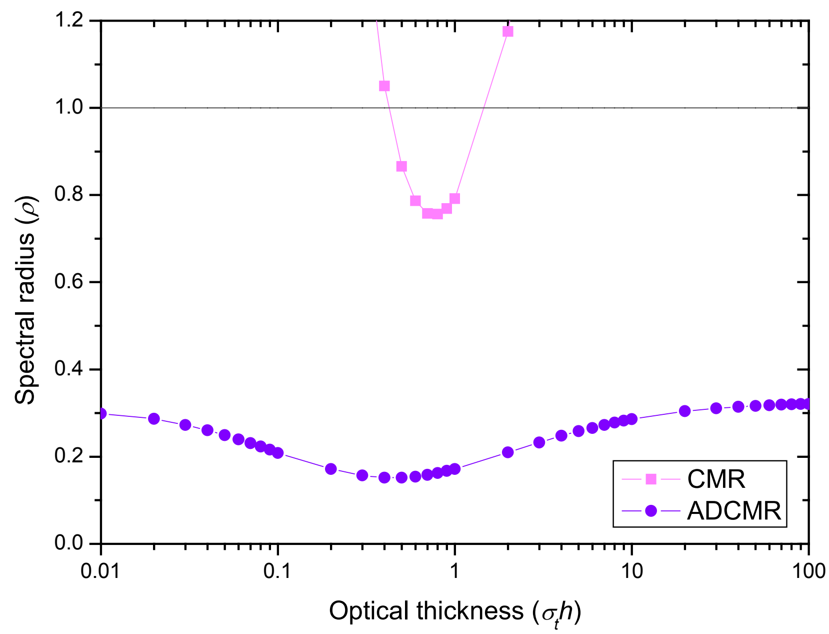

Figure 10. CMR vs ADCMR (DD, *p=2*, *c=1*).

5. CONCLUSIONS

In this paper, a new coarse-mesh nonlinear iteration method based on the rebalance factor concept that is angular dependent and defined only on the coarse-mesh boundaries, called the Angular Dependent Coarse-Mesh Rebalance (ADCMR) method, was developed to accelerate one-dimensional discrete ordinates transport calculations. The ADCMR method was successfully tested on several benchmark problems. Also, as a theoretical analysis of the ADCMR method, the Fourier analysis through linearization was performed. The results show that the ADCMR method is very effective and unconditionally stable in terms of the spectral radius and the number of iterations. Furthermore, the spectral radius decreases as the coarseness increases. We conclude that extensions of the ADCMR method to multi-dimensional and realistic transport calculations should be feasible and that, in view of the encouraging results above, the method warrants its continued development.

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REFERENCES

1. H. J. Kopp, "Synthetic Method Solution of the Transport Equation," *Nucl. Sci. Eng.*, **17**, 65 (1963); see also *Synthetic Method Solution of the Neutron Transport Equation*, PhD Thesis, University of California at Berkeley (1962).
2. G. I. Marchuk and V. I. Lebedev, *Numerical Methods in the Theory of Neutron Transport*, Second Revised Edition, Harwood Academic Publishers, London (1986). [Originally published (1971) in Russian by Isdatel'stvo Atomizdat, Moscow.]
3. R. E. Alcouffe, "Diffusion Synthetic Acceleration Methods for the Diamond Differenced Discrete Ordinates Equations," *Nucl. Sci. Eng.*, **64**, 344 (1977).
4. E. W. Larsen, " Unconditionally Stable Diffusion Synthetic Acceleration Methods for the Slab Geometry Discrete-Ordinates Equations, Part I : Theory," *Nucl. Sci. Eng.*, **82**, 47 (1982).
5. D. R. McCoy and E. W. Larsen, "Unconditionally Stable Diffusion Synthetic Acceleration Methods for the Slab Geometry Discrete-Ordinates Equations, Part II : Numerical Results," *Nucl. Sci. Eng.*, **82**, 64 (1982).
6. G. L. Ramone and M. L. Adams, "A Transport Synthetic Acceleration Method for Transport Iterations," *Nucl. Sci. Eng.*, **125**, 257 (1997).
7. V. Ya. Gol'din, "A Quasi-Diffusion Method of Solving the Kinetic Equation," Zh. Vych. Mat., 4(1078), 1964, English Translation published in USSR Comp. Math. And Math. Phys., 4, 136 (1967).
8. E. E. Lewis and W. F. Miller, Jr., "A Comparison of P_1 Synthetic Acceleration Techniques," *Trans. Am. Nucl. Soc.*, **23**, 202 (1976).

9. G. R. Cefus and E. W. Larsen, “Stability Analysis of the Quasidiffusion and Second Moment Methods for Iteratively Solving Discrete Ordinates Problems,” *Transport Theory and Statist. Phys.*, **18**, 493 (1990).
10. T. A. Germogenova, “Convergence of Some Approximate Methods of Solving the Transport Equation,” *Soviet Math. Dokl.*, **9**, No. 4, 855, (1969).
11. D. Y. Anistratov and V. Ya. Gol’din, “Nonlinear Methods for Solving Particle Transport Problems,” *Transport Theory and Statist. Phys.*, **22**, 125 (1993).
12. D. Y. Anistratov and E. W. Larsen, “Alpha-Weighted Acceleration Methods for the Transport Equation,” *Trans. Am. Nucl. Soc.*, **75**, 154 (1996).
13. M. L. Adams and E. W. Larsen, “Fast Iterative Methods for Discrete-Ordinates Particle Transport Calculations,” *Prog. Nucl. Energy*, **40**, 3 (2002)
14. J. J. Duderstadt and W. R. Martin, *Transport Theory*, John Wiley & Sons, 1976.
15. G. R. Cefus and E. W. Larsen, “Stability Analysis of Coarse-Mesh Rebalance,” *Nucl. Sci. Eng.*, **105**, 31 (1990).
16. S. G. Hong and N. Z. Cho, “Rebalance Approach to Nonlinear Iteration for Solving the Neutron Transport Equations,” *Ann. Nucl. Energy*, **24**, 147 (1997).
17. S. G. Hong and N. Z. Cho, “Angle-Dependent Rebalance Factor Method for Nodal Transport Problems in X-Y Geometry,” *Trans. Am. Nucl. Soc.*, **79**, 139 (1998).
18. C. J. Park and N. Z. Cho, “Additive Angular Dependent Acceleration Method of the Discrete Ordinates Transport Calculations,” *ANS International Meeting on Mathematical Methods for Nuclear Applications (M&C 2001)*, September 2001, Salt Lake City, U.S.A. (CD-ROM)
19. C. J. Park and N. Z. Cho, “Convergence Analysis of Additive Angular Dependent Rebalance Acceleration for the Discrete Ordinates Transport Calculations,” *Nucl. Sci. Eng.*, **142**, 64 (2002).
20. M. L. Adams, “New Nonlinear Method for Linear Transport Calculations,” *Proc. ANS Topical Meeting, Mathematical Methods and Supercomputing in Nuclear Applications, M&C + SNA '93*, April 19-23, 1993, Karlsruhe, Germany, **1**, 683.
21. M. L. Adams and W. R. Martin, “Boundary Projection Acceleration : A New Approach to Synthetic Acceleration of Transport Calculation,” *Nucl. Sci. Eng.*, **100**, 177 (1988).