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PROCEDURE TO OBTAIN THE ELEMENTARY SOLUTIONS OF THE MONOENERGETIC EQUATION OF NEUTRON TRANSPORT FOR NON-PLANE GEOMETRIES

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ABSTRACT

The keystones of the procedure to obtain the solutions of the mono-energetic equation of neutron transport for non-plane geometries are presented. This procedure was described for a long time [1] but only in Russian books and papers. Non-plane elementary solutions themselves were published in many reports [2], papers in English [3], in book [4]. Numerous applications of these solutions were described in international conference's reports and in the paper [5] through solving of reactor problems where the Green's functions and Surface Pseudo Source Method were put into practice. However a number researchers expressed a natural desire to acquaint oneself with the procedure to obtain of non-plane elementary solutions so far as there are not trivial points in it and as Russian language is uncommon one for the most reactor's specialists.

Key words: neutron, transport, solution, equation, geometry

1. INTRODUCTION

During 60-th years after the work of K.M. Case [6] a great number of works was devoted to the study and the use of elementary solutions of neutron transport equation. (The detailed bibliography there is in [7]. Most of the works dealed with the plane geometry. To extend the method to the problems in the non-plane geometry prove to be not trivial. To solve such problems there were suggested different methods permitting to dispense with the construction of elementary solutions in such geometries, and reducing the problem to the solution of some auxiliary plane tasks [8-12]. Since the study of elementary solutions helps to understand behaviour of solutions of transport equation the investigation of their properties in non-plane geometries and derivation of their explicit form is of certain interest, in our opinion.

In this paper we give some results concerning elementary solutions of transport equation in the non-plane geometries, following basically the works [1].

2.OVERVIEW AND KEYSTONES OF PROCEDURE

Let us start with a few general remarks.
We shall consider the stationary transport equation for monoenergetic neutrons in the homogeneous medium:

$$\vec{\Omega}\vec{\nabla}\Psi\left(\vec{r},\vec{\Omega}\right)+\Psi\left(\vec{r},\vec{\Omega}\right)=\frac{c}{4p}\int f\left(\vec{\Omega}\vec{\Omega}'\right)\Psi\left(\vec{r},\vec{\Omega}'\right)d\vec{\Omega}' \tag{1}$$

Here, $f\left(\vec{\Omega}\vec{\Omega}'\right)$ is the scattering kernel normalized per unity. The rest symbols are usual. It is known [13], that when finding approximate solutions with the spherical harmonics method, it is possible to re-write the system of equations for N-angular moments of distribution function excluding all moments except zero-order one. In this case, we obtain the next equation:

$$\left[\prod_{i}^{\frac{1}{2}(N+1)}\left(\nabla^2-\frac{1}{\boldsymbol{n}_j^2}\right)\right]\Psi_o\left(\vec{r}\right)=0$$

Therefore, the general solution for $\Psi_o\left(\vec{r}\right)$ is the sum

$$\Psi_o\left(\vec{r}\right)=\sum_i A_i\Psi_o^i\left(\vec{r}\right) \tag{2}$$

where $\Psi_o^i\left(\vec{r}\right)$ is the solution of equation:

$$\left(\nabla^2-\frac{1}{\boldsymbol{n}_i^2}\right)\Psi_o^i\left(\vec{r}\right)=0 \tag{3}$$

The spectrum of $\boldsymbol{n}_i$ is discrete, and depend on the approximation number but independent on the geometry.

One can hope that while finding the exact solution the flux $\Psi_o\left(\vec{r}\right)$ will be as before represented as a superposition of functions each of which is the solution of equation (3). much as the part of spectrum $\boldsymbol{n}_i$ can turn to be continuous, the appropriate part of the sum in (2) will go convert into the integral.

The arbitrary solution of equation (3) at the fixed $\boldsymbol{n}_i$ dependents on two more parameters. For instance, in Cartesian coordinates it has the form:

$$\Psi_o^{imn}\left(\vec{r}\right)=\exp\pm\left(\frac{m}{\boldsymbol{n}_i}x+\frac{n}{\boldsymbol{n}_i}y+\frac{p}{\boldsymbol{n}_i}z\right)$$

where $m^2+n^2+p^2=1$.

Otherwise the solution can be written as

$$\Psi_o^i(\vec{r},\vec{w}) = \exp\left(\pm \frac{(\vec{r}\vec{w})}{n_i} \right) \tag{4}$$

where $\vec{w}$ is the unit length vector with the components $w_z = \cos J = p$, $w_x = \sin J \cos a = m$, $w_y = \sin J \sin a = n$.

Note, that J and a can be complex number (**it is important**). Indeed, the condition

$m^2 + n^2 + p^2 = \cos^2 J + \sin^2 J(\cos^2 a + \sin^2 a) = 1$

is valid at any complex number J and a.

For each $\Psi_o^i(\vec{r},\vec{w})$ one can find the respective elementary solution $\Psi^i(\vec{r},\vec{w},\vec{\Omega})$. To do this firstly we substitute $\Psi_o^i(\vec{r},\vec{w})$ (with the arbitrary n_i) into the right-hand side of equation (1) and believing that it is known we shall solve the equation. The solution of the appropriate homogeneous equation $\vec{\Omega}\vec{\nabla}\Psi(\vec{r},\vec{\Omega}) + \Psi(\vec{r},\vec{\Omega}) = 0$ may be added to this solution. Solutions of the last one easy to find. Then, asking for conditions:

$$\int \Psi^i(\vec{r},\vec{w},\vec{\Omega})d\vec{\Omega} = \Psi_o^i(\vec{r},\vec{w})$$

to be valid, we shall determine the spectrum of n_i.

Elementary solutions that or other symmetry can be obtained from the solutions $\Psi^i(\vec{r},\vec{w},\vec{\Omega})$ by means of integration with the corresponding function $A(\vec{w})$, that is

$$\Psi^i(\vec{r},\vec{\Omega}) = \int A(\vec{w})\Psi^i(\vec{r},\vec{w},\vec{\Omega})d\vec{w} \tag{5}$$

So, for spherical symmetric solutions it is necessary to take $A(\vec{w})$=1, for cylindrical symmetric ones $A(\vec{w}) = d(J - p/2)$.

Note, that the expression (5) reminds "non-plane" elementary solutions considered in the works /4,10,11/ which were constructed from the "plane" ones. Indeed, solutions (4) with real vector $\vec{w}$ represent "plane" elementary solutions in the coordinate system which orientation is determined by the vector $\vec{w}$. But the fact is significant essential that in the mentioned works the unit vector $\vec{w}$ was regarded as real one whereas the application of vectors with complex components is needed. The question on choice of integration ways in the complex planes J and a is raised.

In works [8-12] only plane elementary solutions were used in expressions of type (5). Such a limitation seems to be inconvenient. We shall use the extended system of plane elementary solutions [1]. (See also [14], appendix J).

For one-velocity equation with isotropic scattering such solutions have the form:

$$\Psi^{m}(\nu, x, \mu, \alpha) = \exp\left(-\frac{x}{\nu}\right)\Phi_{m}(\nu, \mu)\exp(im\alpha)$$

$$\Phi_{m}(\nu,\mu) = \begin{cases} \frac{c\nu}{2} P\frac{1}{\nu-\mu} + \lambda(\nu)\delta(\nu-\mu), m = 0, \nu \in [-1,1] \\ \delta(\nu-\mu), m \neq 0 \end{cases} \quad (6)$$

Here α is the azimuthally angle, that is, the angle between $(\vec{\Omega}, \vec{x})$ and $(\vec{x}, \vec{y})$. The spectrum ν is known.

While substituting these solutions in (5) and carrying out the integration by $\vec{w}$, considering that $\vec{w}$ is real vector we shall obtain the system of regular elementary solutions. When $A(\vec{w}) = 1$ the solutions having the spherical symmetry is described as

$$\varphi(\nu, r, \mu) = \sum_{n=0}^{\infty}\frac{2n+1}{2}(-1)^{n}A_{n}(\nu)\frac{I_{n+\frac{1}{2}}\left(\frac{r}{\nu}\right)}{\sqrt{r}}P_{n}(\mu) \quad (7)$$

Here $P_{n}(\mu)$ are Legendre polynomials $I_{m}(x)$ are Bessel' functions of a imaginary argument, $A_{n}(\nu)$ are polynomial on n-order.

$$A_{n}(\nu) = \int_{-1}^{+1}\Phi_{o}(\nu,\mu)P_{n}(\mu)d\mu.$$

With $A(\vec{w}) = \delta\left(\vartheta - \frac{\pi}{2}\right)$ we get cylindrical symmetrical elementary solutions:

$$\varphi^{m}(\nu, r, \theta, \alpha) = \sum_{n=m}^{\infty}\frac{2n+1}{2}A_{n}^{m}(\nu)\sum_{k=0,2}^{\alpha\left[\frac{n}{2}\right]}\Phi_{k,m}^{n}\left(\frac{r}{\nu}\right)Y_{n}^{k}(\theta,\alpha) \quad (8)$$

Here $q = \arccos\left(\frac{\vec{\Omega}\vec{r}}{|\vec{r}|}\right), m = \cos q$, α is the angle between planes $\left(\vec{\Omega}\vec{r}\right)$ and $\left(\vec{x}, \vec{y}\right)$, r is the radius-vector in the plane $\left(\vec{x}, \vec{y}\right)$ drawn to the point of neutron location, Y_n^k are spherical functions.

$$A_n^m(v) = \begin{cases} A_n(v), & m = 0 \\ \frac{\sqrt{(n-m)!}}{(n+m)!} P_n^m(v), & m \neq 0 \end{cases}$$

$P_n^m(v)$ are the associated Legendre functions; $\Phi_{k,m}^n\left(\frac{r}{v}\right)$ are functions which are expressed through Bessel functions $I_\ell\left(\frac{r}{v}\right)$.

The functions (7) and (8) are insufficient to solve the corresponding spherical and cylindrical problems. It can be seen than among them there are no functions decreasing at r→∞. Additional solutions are obtained by the method [1] being similar to that which are applied to get divergent spherical and cylindrical waves while solving the wave equation (see, for example, [15]).
As a result we obtain:

- for spherical geometry

$$\Psi(v,r,m) = \sum_{n=0}^{\infty} \frac{2n+1}{2} A_n(v) \frac{K_{n+\frac{1}{2}}\left(\frac{r}{v}\right)}{\sqrt{r}} P_n(m) \tag{9}$$

- for cylindrical geometry

$$\Psi^m(v,r,q,a) = \sum_{n=m}^{\infty} \frac{2n+1}{2} A_n(v) \sum_{k=0,2}^{2\left[\frac{n}{2}\right]} F_{k,m}^n\left(\frac{r}{v}\right) Y_n^k(J,a) \tag{10}$$

Here $K_m(x)$ are the modified Bessel functions; $F_{k,m}^n\left(\frac{r}{v}\right)$ are functions expressed through $K_\ell\left(\frac{r}{v}\right)$.

The series written at the expressions (9) and (10) are strongly divergent (**it is important**), since $K_m(x) \approx \frac{1}{2}\Gamma(m)\left(\frac{2}{x}\right)^m$, μ→∞; $\Gamma(m)$ is the gamma-function). One can give a sense to them if they are considered as generalized functions (distribution) to which very smooth basic functions correspond. In this case, the functions (9) and (10) are solutions of the transport equation everywhere except the origin.

While considering limits which substitute operations with such functions, solutions obtaining in P_N-approximations of the spherical harmonics method are used. The convergence of solutions in P_N-approximation to the exact with $N \to \infty$ leads to conclusion that the system of elementary solutions is complete.

Conditions of biorthogonality to elementary solutions of adjoint equation are valuable for the use of the elementary solutions.

3.THE ORTHOGONALITY OF ELEMENTARY SOLUTIONS

Firstly, let us obtain one general expression. Let us consider homogeneous transport equation:

$$\vec{\Omega}\vec{\nabla}\Psi_x(\vec{r},\Omega,e)+\Sigma(e)\Psi_x(\vec{r},\vec{\Omega},e)=\int K(e,e',\vec{\Omega}\vec{\Omega}')\Psi_x(\vec{r},\vec{\Omega}',e')de'd\vec{\Omega}'$$

and some set of its solutions $\Psi_x(\vec{r},\vec{\Omega},e)$. Here, ε is neutron energy; Σ(ε) is total neutron cross-section; $K(e,e',\vec{\Omega},\vec{\Omega}')$ is kernel which can take into account both scattering and multiplication of neutrons.

Let us write the adjoin equation

$$-\vec{\Omega}\vec{\nabla}\Psi^+_{x'}+\Sigma(e)\Psi^+_{x'}=\int K(e',e,\vec{\Omega}\vec{\Omega}')\Psi^+_{x'}(\vec{r},\vec{\Omega}',e')de'd\vec{\Omega}'$$

with its set of solutions $\Psi^+_{x'}(\vec{r},\vec{\Omega},e)$. If $\Psi_x(\vec{r},\vec{\Omega},e)$ and $\Psi^+_{x'}(\vec{r},\vec{\Omega},e)$ are such that there are integrals $\vec{j}=\int\vec{\Omega}\Psi_x\Psi^+_{x'}d\vec{\Omega}'de$, multiplying the direct equation by $\Psi^+_{x'}$ and the adjoin equation by $\Psi_{x'}$, integrating over ε and over $\vec{\Omega}$ and subtracting the second equation from the first one, let find:

$$div_{\vec{r}}\,\vec{j}(x,x',\vec{r})=0 \tag{11}$$

If we deal with one-velocity equation, at this relation $\vec{j}=\int\vec{\Omega}\Psi_x\Psi^+_{x'}d\vec{\Omega}'$.

Let us note that the expression (11) will be valid not only for the exact solutions of transport equation, but for approximate ones, obtained on P_N-approximations of spherical harmonics method. Indeed, if Ψ_x^N is some solution of homogeneous equation obtained in P_N-approximation and, consequently, including spherical harmonics up to N-order, the result of action of the transport operator $\mathbf{K}$ on this solution will include only spherical harmonics of N+1-order:

$$\mathbf{K}\Psi_x^N\left(\vec{r},\vec{\Omega},e\right)=\sum_{m=0}^{N}a_m \mathrm{Y}_{N+1}^m\left(\vec{\Omega}\right)$$

Similar statement is also valid for the adjoin equation:

$$\mathbf{K}\Psi_x^{1,N}\left(\vec{r},\vec{\Omega},e\right)=\sum_{m=0}^{N}a_m^+ \mathrm{Y}_{N+1}^m\left(\vec{\Omega}\right)$$

Therefore, multiplying the direct equation by $\Psi_{x'}^{+N}$ and the adjoin equation by Ψ_x^N and integrating over $\vec{\Omega}$, we obtain zero on the right - hand side using of orthogonality of spherical functions. In the operator part there remains again the expression:

$$div_{\vec{r}}\ \vec{j}^{\,N}(x,x',r)=0 \tag{12}$$

where $\vec{j}^{\,N}(x,x',r)=\int\vec{\Omega}\Psi_x^N\Psi_{x'}^{+N}d\vec{\Omega}de$.

The relations (11) and (12) can be used to prove the orthogonality of elementary solutions. Let us deal, for instance, with the plane one-velocity problem with. isotropic scattering. Elementary solutions are determined by equations (6), corresponding adjoint solutions are derived from there equations with substituting $\mu\to-\mu,\ \alpha\to\alpha+\pi$. The equation (11) will take the form

$$\frac{\partial}{\partial x}j(v,m,v',m',x)=0\,.$$

Therefore, $j(v,m,v',m',x)=c(v,m,v',m')$ is independent on x. But

$$j(v,m,v',m',x)=\exp\left(-\frac{x}{v}+\frac{x}{v'}\right)\iint d\mu d\alpha\Phi_m(v,\mu)\Phi_{m'}(v',\mu)\exp\left(i(m+m')\alpha\right).$$

Therefore, $c(v,m,v',m')=0$ when $v\neq v'$.

Let us consider now an equation for cylindrical symmetry problems. In this case, the equation (11) and (12) take the form, responsively:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rj(x,x',r)\right]=0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rj^N(x,x',r)\right]=0$$

Hence,

$$j(x, x', r) = \frac{c(x, x')}{r}$$

$$j^N(x, x', r) = \frac{c^N(x, x')}{r} \qquad (13)$$

Let's take elementary solutions determined by the formulae (8) and (10) and the corresponding solutions of adjoin equation. Since in expression (13) $c(x,x') = c(v,m,x,v',m',x')$, $c^N(x,x') = c^N(v,m,x,v',m',x')$ are independent on r, it is enough to define them in one point, for example, at r→∞ (with the symbol ξ we shall record a nature of elementary solutions, assuming that ξ=0 for regular solutions and ξ=1 for singular ones).

Taking into account the asymptotic behavior of elementary solutions we obtain the orthogonality of elementary solutions $j^m(v,r,q,a)$ and respective adjoin elementary solutions $[\boldsymbol{j}^m(v',r,\boldsymbol{q},\boldsymbol{a})]^+ = \boldsymbol{j}^m(v',r,\boldsymbol{p}-\boldsymbol{q},\boldsymbol{a}+\boldsymbol{p})$ at all v, m, v', m'.

The case is more complicated with singular solutions, since for them there are no $j(v,m,x,v',m',x')$ in a common sense. But this value can be considered as some generalization of convolution operation of generalized functions.
Defining it as follows:

$$j(x,x',r) = \lim_{N\to\infty} j^N\left(v^N, m, x, v'^N, m', x'\right), \qquad (14)$$

where v^N and v'^N are the nearest to v and v', respectively, roots of secular equation in P_N-approximation of spherical harmonics method and using once again the asymptotic behavior of elementary solutions, we write:

$$j(v,m,x,v',m',x') = \frac{vN(v,m)}{4r} d\,(v,m,v',m')(1 - dxx') \qquad (15)$$

The orthogonality of spherical elementary solutions is substantiated in an analogous way.

$$d\,(v,m,v',m') = \begin{cases} 0 & v \neq v', m \neq m' \\ & d\,(v - v'), m = m' \end{cases}$$

4. GREEN'S FUNCTIONS

The established conditions of biorthogonality allow to obtain, in the standard way, Green functions for infinite homogeneous medium with sources of definite symmetry. Expressions for these Green functions were used in the surface pseudosources method [16,17,18]. Let's write, for example, Green function corresponding to the source $\dfrac{\boldsymbol{d}(r-R)}{2\boldsymbol{p}R^2}$

$$\boldsymbol{j}^{is}(r,m/R)=\begin{cases}\dfrac{I_{1/2}\left(\dfrac{R}{v_o}\right)\Psi(v_o,r,m)}{2\boldsymbol{p}\sqrt{R}v_oN(v_o,0)}+\displaystyle\int_0^1\dfrac{I_{1/2}\left(\dfrac{R}{v}\right)\Psi(v,r,m)dv}{2\boldsymbol{p}\sqrt{R}vN(v,0)},r>R\\ \dfrac{K_{1/2}\left(\dfrac{R}{v_o}\right)\boldsymbol{j}(v_o,r,m)}{2\boldsymbol{p}\sqrt{Rv_oN(v_o,0)}}+\displaystyle\int_0^1\dfrac{K_{1/2}\left(\dfrac{R}{v}\right)\boldsymbol{j}(v,r,m)dv}{2\boldsymbol{p}\sqrt{RvN(v,0)}},r<R\end{cases}\tag{16}$$

At r<R, the solution is expressed through ordinary functions, whereas at r>R we have the linear combination of generalized functions. But on the order hand $\boldsymbol{j}^{is}(r,m,R)$ can be expressed as integrals $\boldsymbol{j}_o^{is}(r/R)$ function represented by the expression:

$$\boldsymbol{j}_o^{is}(r/R)=\int_{-1}^{+1}\boldsymbol{j}^{is}(r,m/R)dm$$

$$j_o^{is}(r/R)=\begin{cases}\dfrac{\exp\left(\dfrac{R-r}{v_o}\right)-\exp\left(\dfrac{-R+r}{v_o}\right)}{RrN(v_o,0)}+\displaystyle\int_0^1\dfrac{\exp\left(\dfrac{R-r}{v}\right)-\exp\left(\dfrac{-R+r}{v}\right)}{RrN(v,0)}dv,r>R\\ \dfrac{\exp\left(\dfrac{-R-r}{v_o}\right)-\exp\left(\dfrac{-R+r}{v_o}\right)}{RrN(v_o,0)}+\displaystyle\int_0^1\dfrac{\exp\left(\dfrac{-R-r}{v}\right)-\exp\left(\dfrac{-R+r}{v}\right)}{RrN(v,0)}dv,r<R\end{cases}$$

$$\boldsymbol{j}_o^{is}(r/R)=\int_0^{\infty}\boldsymbol{j}_o^{is}(\vec{r}-\boldsymbol{h}\vec{\Omega})\exp(-\boldsymbol{h})d\boldsymbol{h}+S(r,m)$$

Here S(r,μ) is contribution neutrons came into the point from the source without collision. For the isotropic source S(r,μ) is a ordinary bounded function for all μ and almost for all r expect r=R. Also an integral of $\boldsymbol{j}_o^{is}(r/R)$ exists almost always, since the integrand is the fast-decreasing at the infinity and almost everywhere bounded function, expect the point r=R, where the integrand has the logarithmic singularity which does not impede the integral to exist.

From this it follows that $j^{is}(r, m/R)$ has to be an ordinary function bounded almost at all r. Consequently, generalized functions go into the expression (16) that distant terms in sums (16) responsible for their divergence are mitially compensated. Such a compensation, is possible only at the simultaneous consideration of parts solution corresponding to discrete and continuous spectrum of ν*⁾.
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The circumstance is in agreement with Davison'note [13] that for the problems with spherical symmetry problems with at any finite distance from the source in order to get the angular moments of distribution function with the fairly high number N, the transient part of the solution cannot be dropped.

From the mentioned above it is clear that the solution in term of two parts (asymptotic and transient) is suited for calculation of angular moments of distribution function but is not very convenient for seeking the number of neutrons moving at a definite angle. But note, that we have to deal with the first problem more often than with the second one.

There was reported above on elementary solutions having the spherical and cylindrical symmetry. But likewise, more general elementary solutions can be constructed, in particular, such solutions which are needed to construct Green functions for the monodirectional point or monodirectional linear sources. The way to get solutions is the following:

1. The system of non-plane regular elementary solutions is obtained from the plane elementary solutions with the help if integrals of the form

$$j^{m,k}_{v,p}\left(\vec{r},\vec{\Omega}\right) = \int_{4p} d\vec{w}\,Y^{k}_{p}(\vec{w})\Psi^{m}\left(v, \frac{\vec{r}\vec{w}}{r}, \vec{\Omega}\vec{w}, dw\right) Y^{k}_{p}\left(\frac{\vec{r}}{r}\right)$$

for the spherical coordinate system and

$$j^{m}_{v,p}\left(\vec{r},\vec{\Omega}\right) = \int_{0}^{2p} dx\, \exp\left(ip(x - x_0)\right)\Psi^{m}\left(v, \frac{\vec{r}\vec{w}}{r}, \vec{\Omega}\vec{w}, dw\right)\Bigg|_{w_z = 1}$$

for the cylindrical coordinate system (for two-dimensional problem).

2. The system of singular elementary solutions is found from the plane solutions at the following integrating

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*⁾ To show in detail how such a compensation takes place, it is possible using the orthogonality of polynomials $A_n(\nu)$ (see [1] appendix 7b).

$$\Psi_{v,p}^{m,k}\left(\vec{r},\vec{\Omega}\right)=\int_{0}^{2p}da\int_{0}^{i\infty}dm\Upsilon_{p}^{k}\left(\vec{w}\right)\Psi^{m}\left(v,\frac{\vec{r}\vec{w}}{r},\vec{\Omega}\vec{w},dw\right)\Upsilon_{p}^{k}\left(\frac{\vec{r}}{r}\right)$$

and

$$\Psi_{v,p}^{m}\left(\vec{r},\vec{\Omega}\right)=\int_{0}^{i\infty}dx\,\exp\left(ip(x-x_0)\right)\Psi^{m}\left(v,\frac{\vec{r}\vec{w}}{r},\vec{\Omega}\vec{w},dw\right)_{w_z=1}$$

for the spherical and cylindrical coordinate system, respectively.

In this case, there arises the question on extension of plane elementary solutions in complex plane. Such an extension is very simply done if the plane elementary solutions are written in term of spherical harmonics expansion.

3. With the help of equation (11) being integrated by directions of radius-vector **r**, and knowing the asymptotic behavior of elementary solutions, we get (ortho-normalized) relations for the system of elementary solutions and adjoin solutions.

4. Having the ortho-normalized relations, find Green function and any its moments. et us emphasize the fact that the plane elementary solutions in any case can be written as:

$$\Psi^{m}\left(v,x,\vec{\Omega},e\right)=\exp\left(-\frac{x}{v}\right)\sum_{n=m}^{\infty}a_{n}^{m}\left(v,e\right)\Upsilon_{n}^{m}\left(\Omega\right),$$

where all information on the energetic and angular dependence is contained in coefficients $a_n^m(v,e)$. But when passing from the plane geometry to any other one all transformations are merely concerned with expressions $\exp\left(-\frac{x}{v}\right)\Upsilon_{n}^{m}\left(\vec{\Omega}\right)$.

Therefore, with such approach it is easy to use the results obtained for the plane problem if taking into account the scattering anisotropy, energetic dependence e.t.a.

5.GENERAL FORM GREEN'S FUNCTIONS FOR THREE GEOMETRIES

Represent the Green function G_{inf} as the decomposition:

$$G\left(\vec{x},\vec{x}',\vec{\Omega},\vec{\Omega}'\right)=\sum_{n}\sum_{n'}\sum_{m}\sum_{m'}G_{nn'}^{mm'}Y_{nm}\left(\vec{\Omega}\right)Y_{n'm'}\left(\vec{\Omega}'\right)$$

For problems with plane, spherical and cylindrical symmetries we have:

$$G_{nn'}^{mm'}(\vec{x},\vec{x}') = \begin{cases} \sum_{l=0}^{L} \underset{n}{S} \dfrac{A_n^l(n) A_{n'}^l(n) X_{nm}^l\left(\dfrac{(\vec{x},\vec{n})}{n}\right) Z_{n'm'}^l\left(\dfrac{(\vec{x}',\vec{n}')}{n}\right)}{N(n,l)}; (\vec{x},\vec{n}) \geq (\vec{x}',\vec{n}') \\ \sum_{l=0}^{L} \underset{n}{S} \dfrac{A_n^l(n) A_{n'}^l(n) X_{nm}^l\left(\dfrac{(\vec{x}',\vec{n}')}{n}\right) Z_{n'm'}^l\left(\dfrac{(\vec{x},\vec{n})}{n}\right)}{N(n,l)}; (\vec{x},\vec{n}) \leq (\vec{x}',\vec{n}') \end{cases}$$

Here L = min(n, n')

$\underset{n}{S} f(n) = \int_0^1 f(n)\,dn + \sum_{\{n_k\}} f(n_k)$ {ν k}- the positive part of discrete spectra of eigenvalues;

$$A_n^l(n) = \begin{cases} P_n^l(n) & l \neq 0 \\ P_n^l(n) - cn\left[Q_0(n)P_n(n) - Q_n(n)\right] & l = 0 \end{cases}$$

$$N(n, l) = \begin{cases} n & l \neq 0 \\ \left[l^2(n) + \dfrac{p^2 c^2}{4} n^2 \right] n & n \in [-1,1], \quad l = 0 \\ \dfrac{c n_0^3}{2}\left(\dfrac{c}{n_0^2 - 1} - \dfrac{1}{n_0^2} \right) & n = n_0, \quad l = 0 \end{cases}$$

For planar geometry:

$$X_{nm}^l = d_{m0} d_{l0} \exp\left(-\frac{x}{n}\right) \qquad Z_{nm}^l = d_{m0} d_{l0} \exp\left(\frac{x}{n}\right)$$

For spherical geometry:

$$X_{nm}^l = d_{m0} d_{l0} \frac{K_{n+1/2}(r/n)}{\sqrt{rn}}; \qquad Z_{nm}^l = d_{m0} d_{l0} \frac{I_{n+1/2}(r/n)}{\sqrt{rn}}$$

For cylindrical geometry:

$$X_{nm}^l = d_{m,2p} d_{l,2p} F_{ml}^n\left(\frac{r}{n}\right) n^{-\frac{1}{2}}; Z_{nm}^l = d_{m,2p} d_{l,2p} \Phi_{ml}^n\left(\frac{r}{n}\right) n^{-\frac{1}{2}}$$

$$F_{ml}^n\left(\frac{r}{n}\right) = \sum_{k=0}^{n} a_{mk}^{nl} K_k\left(\frac{r}{n}\right); \Phi_{ml}^n\left(\frac{r}{n}\right) = (-1)^n \sum_{k=0}^{n} a_{mk}^{nl} I_k\left(\frac{r}{n}\right)$$

The functions K_n (x) and I_n (x) are the modified Bessel functions. The coefficients $a_{m\,k}^{n\,l}$ are given in [1]. $\vec{n},\vec{n}'$ are the unit normals to the surfaces of constant r and r' respectively.

For one velocity problems with isotropic scattering:

5. ELEMENTARY SOLUTIONS OF THE TRANSPORT EQUATION FOR CYLINDRICAL GEOMETRY WITH THE AZIMUTHAL ASYMMETRY

The Green function of the transport equation for the homogeneous infinite medium is required for the surface pseudosources method [16-24], which is especially attractive for solving the one-speed transport equation in the nuclear reactor cells. This one-speed Green function is convenient to present by means of the elementary solutions of the one-speed transport equation. The elementary solutions for an arbitrary geometry are obtained from the elementary solutions of the plane geometry transport equation. So the elementary solutions of the one-speed transport equation for problems with the spherical and cylindrical symmetries and for problems in the cylindrical geometry with azimuthal asymmetry were obtained. The elementary solutions of the one-speed transport equation for the cylindrical geometry with azimuthal asymmetry were used. A set of the elementary cylindrical solutions for an arbitrary symmetry was obtained in this part. This system may be used for the 2-dimensional problems.

The transport equation in the cylindrical coordinate system for a homogeneous medium is:

$$\mu\frac{\partial\Psi(r,\alpha,\mu,\varphi)}{\partial r}+\frac{(1-\mu^2)\cos^2\varphi}{r}\frac{\partial\Psi}{\partial\mu}+\frac{\mu\sin^2\varphi}{2r}\frac{\partial\Psi}{\partial\varphi}+\frac{(1-\mu^2)^{\frac{1}{2}}\cos\varphi}{r}\frac{\partial\Psi}{\partial\alpha}+\Psi=\frac{c}{4\pi}\int_{4\pi}\Psi(r,\alpha,\vec\Omega')d\vec\Omega' \qquad (17)$$

where $\Psi(r,\alpha,\mu,\varphi)$ is the neutron distribution function; $\vec\Omega$ is the vector along the neutron direction; $\mu=\cos\theta$, $(\theta=\arccos(\vec\Omega\vec r))$ is the cosine of the angle between the neutron direction $\vec\Omega$ and the $\vec r$ vector; $\vec r$ is the radius-vector in the (x,y) plane; φ is the angle between the $(\vec\Omega,\vec r)$ plane and the (x,y) plane; α is the azimuthal angle between the $\vec r$-vector and the x-axis; r is measured in units of mean free paths; $c=\Sigma_s/\Sigma_{tot}$, Σ_s is the scattering cross section, Σ_{tot} is the total cross section.

The system of the complex cylindrical regular elementary solutions of the transport equation (17) is obtained from the plane elementary solutions by using the integrals

$$\varphi^{m}_{n,\pi}(\vec r,\vec\Omega)=\int_0^{2\pi}d\zeta\,\exp(i\pi(\xi-\alpha))\Psi^{m}(\nu,(\vec\Omega\vec r),(\vec\Omega\vec\omega,)\alpha_\omega)\Big|_{\omega_z=1} \qquad (18)$$

where $\cos V=(\vec r\vec\omega)$, ν is a parameter, r is the number of the spatial moment of the complex cylindrical regular elementary solution.

After performing some reductions in the expression (18), the real cylindrical regular elementary solutions of the transport equation (17) are obtained in the form

$$f_{n,p}^{m}\left(\vec{r},\vec{\Omega}\right)=\begin{cases}\sum\limits_{n=m}^{\infty}\frac{2n+1}{2}P_{m,0}^{n}(n)\sum\limits_{k=0}^{n}\frac{1}{1+d_{k,0}}\Big\{\left[(-1)^{m}+(-1)^{k}\right]f_{m,k}^{n,p}\left(\frac{r}{n}\right)\cos pa+ \\ \left[(-1)^{m}-(-1)^{k}\right]\tilde{f}_{m,k}^{n,p}\left(\frac{r}{n}\right)\sin pa\Big\}P_{k,0}^{n}\left(\frac{(\vec{\Omega}\vec{r})}{r}\right)\cos kj \\ \sum\limits_{n=m}^{\infty}\frac{2n+1}{2}P_{m,0}^{n}(n)\sum\limits_{k=0}^{n}\frac{-1}{1+d_{k,0}}\Big\{\left[(-1)^{m}+(-1)^{k}\right]f_{m,k}^{n,p}\left(\frac{r}{n}\right)\sin pa- \\ \left[(-1)^{m}-(-1)^{k}\right]\tilde{f}_{m,k}^{n,p}\left(\frac{r}{n}\right)\cos pa\Big\}P_{k,0}^{n}\left(\frac{(\vec{\Omega}\vec{r})}{r}\right)\cos kj \\ \sum\limits_{n=m}^{\infty}\frac{2n+1}{2}P_{m,0}^{n}(n)\sum\limits_{k=0}^{n}\frac{1}{1+d_{k,0}}\Big\{\left[(-1)^{m}+(-1)^{k}\right]\bar{f}_{m,k}^{n,p}\left(\frac{r}{n}\right)\cos pa+ \\ \left[(-1)^{m}-(-1)^{k}\right]\tilde{f}_{m,k}^{n,p}\left(\frac{r}{n}\right)\sin pa\Big\}P_{k,0}^{n}\left(\frac{(\vec{\Omega}\vec{r})}{r}\right)\sin kj \\ \sum\limits_{n=m}^{\infty}\frac{2n+1}{2}P_{m,0}^{n}(n)\sum\limits_{k=0}^{n}\frac{-1}{1+d_{k,0}}\Big\{\left[(-1)^{m}+(-1)^{k}\right]\tilde{f}_{m,k}^{n,p}\left(\frac{r}{n}\right)\sin pa- \\ \left[(-1)^{m}-(-1)^{k}\right]\bar{f}_{m,k}^{n,p}\left(\frac{r}{n}\right)\cos pa\Big\}*P_{k,0}^{n}\left(\frac{(\vec{\Omega}\vec{r})}{r}\right)\sin kj\end{cases}\qquad(19)$$

where

$$f_{m,k}^{n,p}\left(\frac{r}{n}\right)=\frac{(-1)^{p}}{2p}\int_{0}^{2p}\exp\left(-\frac{r\cos x}{v}\right)\left[P_{m,k}^{n}(\cos V)+(-1)^{k}P_{m,-k}^{n}(\cos V)\right]\cos pV dV$$

$$\bar{f}_{m,k}^{n,p}\left(\frac{r}{n}\right)=\frac{(-1)^{p}}{2p}\int_{0}^{2p}\exp\left(-\frac{r\cos x}{v}\right)\left[P_{m,k}^{n}(\cos V)-(-1)^{k}P_{m,-k}^{n}(\cos V)\right]\cos pV dV$$

$$\tilde{f}_{m,k}^{n,p}\left(\frac{r}{n}\right)=\frac{-(-1)^{p}}{2p}\int_{0}^{2p}\exp\left(-\frac{r\cos x}{v}\right)\left[P_{m,k}^{n}(\cos V)-(-1)^{k}P_{m,-k}^{n}(\cos V)\right]\sin pV dV$$

$$\tilde{\bar{f}}^{\,n,p}_{m,k}\left(\frac{r}{n}\right) = \frac{-(-1)^p}{2p}\int_0^{2p} \exp\left(\frac{r\cos x}{v}\right)\left[P^n_{m,k}(\cos V) + (-1)^k P^n_{m,-k}(\cos V)\right]\sin pVdV$$

$P^n_{m,k}(m)$ are the Vilenkin polinomials [25] ($P^n_{m,0}$- are the Legendre polinomials).
The system of the complex cylindrical singular elementary solutions of the transport equation (17) is obtained from the plane elementary solutions by using the integral

$$\Psi^m_{n,p}\left(\vec{r},\vec{\Omega}\right) = \int_0^{i\infty} dz\ \exp\left(ip(x-a)\right)\Psi^m\left(n,(\vec{\Omega}\vec{r}),(\vec{\Omega}\vec{w}),a_w\right)\Big|_{w_z=1} \qquad (20)$$

After performing some reductions in the expression (20), the real cylindrical singular elementary solutions of the transport equation (17) are obtained in the form of the expression (19) with the replacement of the $f^{n,p}_{m,k}\left(\frac{r}{n}\right)$ functions, which are the combination of the Bessel function $I_l\left(\frac{r}{n}\right)$, by the $F^{n,p}_{m,k}\left(\frac{r}{n}\right)$ function, which are the combination of the Bessel function $K_l\left(\frac{r}{n}\right)$ [22,23].
The orthogonality of the regular and singular real cylindrical elementary solutions of equation (17) is established. The normalization coefficients of the elementary solutions (19) and (20) are determined. Using the orthogonality relations for the system of the regular and singular real cylindrical elementary solutions, the Green function of the transport equation (17) is obtained.

REFERENCES

1. .N. I. Laletin 'Elementary solutions of the onevelocity neutron transport equation', charter 7, pp. 155-185 in book 'Methods for Calculations of Thermal Neutron Distributions in Reactor Lattices', Editor Ja. V. Shevelev, Atomisdat, M, 1974
2. N.I.Laletin, Proc. Conf on Physics of Reactors, v. 1, Melekess, (1966), Atomnaya energiya, v. 20, p. 509, (1966); Atomnaya energiya, v. 26, p. 370, (1969).
3. N. I. Laletin, "Comparison of the Surface Pseudosourrces Method (G_N- approximation) with Other Numerical Methods for the Neutron Transport Equation," *Proc. Int. Conf. on Mathematical Methods and Supercomputing for Nuclear Applications*, 5-9 Oct. 1997, p. 945, ANS, Saratoga Springs, New York, USA (1997).
4. I.Marchuk, V. I. Lebedev, 'Numerical Methods in the Theory of Neutron Transport', Harwood Academic Publishers, Cur, p.176, 1986
5. N.I.Laletin, Analysis of the Surface pseudosources method (G_N –approximations) and comparison with orther numerical methods for the neutron transport equation ,TTSP vol 27 n.5-7, p.639-652
6. K.M.Case, Ann. Phys., v. 9,p. 1, (1960).
7. I.Kuscer, N.J.M.McCormick, Advances in Nuclear Science and Thechn. , vol.7 (1973)
8. S.B.Shihov, Yu.J.Yershov. JCM and MPh, v. 12, p. 639, (1972)

9. V.I.Lebedev, JCM and MPh, v. 6, p. 895, (1966)
10. E.Bereiss, AM and MPh, v. 12, p. 3, (1966)
11. G.I.Bell, B.G.Carlson, K.D.Lathrop, Proceeding of the III Geneva Conference, p.261, (1964).
12. G.I.Mitsis, Nucl. Sci. Engng., v. 17, p. 55, (1963).
13. B.Davision, Transport theory of Neutrons, London, (1956)
14. K.M.Case, P.F. Zweifeel, Linear Transport Theory. Michigan, (1967).
15. R.Curant, Equations with partial derivatives. Trasl. from Engl., M., “Mir”, (1964)
16. N.I.Laletin, Preprint IAE-1828, (1969); Atomnaya energiya, v. 28, p. 242, (1970)
17. N.I.Laletin, Preprint IAE-1374, (1967); Proc. “Computed methods in transport theory”, Atomizdat, (1970)
18. N.V. Sultanov, Atomnaya energiya, v. 32, p. 457, (1972)
19. V.F. Boyarinov, Atomnaya energiya, v. 44, p. 250, (1978)
20. Laletin N.I. The Surface Pseudosource Method For Solution Of The Equation Of Neutron Transport (G_n -approximation). Preprint IAE-1374, 1967. In book: The Methods For Calculation Of Thermal Neutron Fields In The Reactor Lattices. Ed. Ya. V. Shevelev, M., Atomizdat, 1974, p.187-215.
21. Laletin N.I., Sultanov N.V. Development of surface pseudosources method for calculation of neutron fields in cells with a cluster of fuel elements. Atomnaya energiya, 46, 148 (1979).
22. Laletin N.I., Sultanov N.V. The Green Function Of The Equation Of Neutron Transport For The Cylindrical Geometry With Azimuthal Asyimmetry. Preprint IAE-2265, M., 1973.
23. Sultanov N.V. Development Of The Surface Pseudosource Method For Calculation Of The Neutron Fields In The RBMK Cells. Preprint IAE-3005, M., 1978.
24. Boyarinov V.F, Development of the surface pseudosources for calculating neutron distribution in a cylindrical reactor cell. Atomnaya Energiya, v59, 2, p.104, 1985
25. Vilenkin N.Y., “Special functions and group presentation”, Nauka, Moskow, 1966 (in. Russian)