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A LINEARIZED THEORY FOR NEAR-EQUILIBRIUM THERMAL RADIATIVE TRANSFER PROBLEMS

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ABSTRACT

The (nonlinear) equations of thermal radiative transfer possess equilibrium solutions of the form $I(\underline{r}, \underline{\Omega}, \nu, t) = B(\nu, T_0)$ and $T(\underline{r}, t) = T_0$, where I is the specific intensity, T is the material temperature, T_0 is any constant temperature, and $B(\nu, T_0)$ is the Planck function at temperature T_0. In this paper, we derive *linearized* radiative transfer equations for problems in which I and T have small deviations from an equilibrium solution, and we present numerical results that demonstrate the accuracy of these equations.

Key Words: thermal radiative transfer

1. INTRODUCTION

In thermal radiative transfer problems [1-9], energy absorbed in a physical system is "emitted" as photons that stream away from their emission sites. The emitted photons (possibly) undergo scattering events and then either leak out of the system or become re-absorbed in the medium at another spatial location. This process is inherently nonlinear, due to the nonlinear dependence on material temperature T of: (i) the Planck function $B(\nu, T)$, which determines the emission rate and frequency-dependence of emitted photons, (ii) the opacity $\sigma(\nu, T)$, which determines the mean free path of the photons, and (iii) the heat capacity $c_v(T)$, which determines the rate of increase in material temperature (as a function of absorbed energy). In this paper, we use linear perturbation theory to derive linearized thermal radiative transfer equations for problems in which the photon and temperature distributions have small deviations from an equilibrium solution. We also provide numerical results that demonstrate the accuracy of the linearized equations.

Various linearizations of the thermal radiative transfer equations have been made previously in order to simulate a nonlinear problem. Such linearizations consist of "freezing" the nonlinear coefficients in the thermal radiative transfer equations during a time step – resulting in a linear problem which is solved for the time step – and then updating the nonlinear coefficients for a subsequent linear calculation for the next time step [1,2,5]. These methods, which have been developed for Monte Carlo [1,2] and deterministic [5] simulations, inherently possess temporal truncation errors. More recently, Su and Olson [7] have shown that for an opacity that is independent of frequency ν and temperature T, and a heat capacity that is proportional to T^3, the radiative transfer equations can be reduced without approximation to a system of linear *grey* (frequency-independent) radiative transfer equations. The linearization developed in this paper is fundamentally different from these previous linearizations, in that (i) we consider a restricted set of *near-equilibrium* problems, but (ii) we make no assumptions on the temperature (or frequency) - dependence of the opacity and heat capacity. The resulting linearized theory applies only to *near-equilibrium* problems, but within this constraint it is universal – it applies to *all* near-equilibrium problems, no matter how nonlinear the original problem is.

To begin, we present the frequency-dependent thermal radiative transfer equations [3,4]. Let $I(\underline{r}, \underline{\Omega}, \nu, t)$ be the specific intensity of radiation (at spatial point $\underline{r}$, traveling in direction $\underline{\Omega}$, with frequency ν, at time t) and $T(\underline{r}, t)$ be the material temperature (at point $\underline{r}$ and time t). Assuming local thermodynamic equilibrium, no scattering, and no material motion, the thermal radiative transfer equations are:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \underline{\Omega} \cdot \underline{\nabla} I = \sigma(B - I) + \frac{Q}{4\pi} \quad , \tag{1.1a}$$

$$c_v \frac{\partial T}{\partial t} = \int_0^\infty \int_{4\pi} \sigma(I - B)\, d\Omega' d\nu' \quad . \tag{1.1b}$$

Our notation is standard: c is the speed of light, $\sigma = \sigma(\nu, T)$ is the opacity, $Q = Q(\underline{r}, \nu, t)$ is an isotropic photon source, and $c_v = c_v(T)$ is the heat capacity. Also,

$$B(\nu, T) = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{1.2a}$$

is the Planck function ($h =$ Planck's constant, $k =$ Boltzmann's constant), which satisfies

$$\int_0^\infty B(\nu, T)\, d\nu = \frac{acT^4}{4\pi} \quad , \tag{1.2b}$$

with $a = 8\pi^5 k^4 / 15 h^3 c^3 =$ radiation constant.

Eqs. (1.1) hold for all $\underline{r} \in D$, (D is the convex spatial system), all unit vectors $|\underline{\Omega}| = 1$, all positive frequencies $0 < \nu < \infty$, and all positive times $t > 0$, where $t = 0$ is the initial time. In general, $\sigma(\nu, T), c_v(T)$ – and, of course, $B(\nu, T)$ – depend nonlinearly on T.

Together with Eqs. (1.1), initial conditions specify I and T at the initial time:

$$I(\underline{r}, \underline{\Omega}, \nu, 0) = I^i(\underline{r}, \underline{\Omega}, \nu) \quad , \tag{1.3a}$$

$$T(\underline{r}, 0) = T^i(\underline{r}) \quad , \tag{1.3b}$$

and a boundary condition specifies I for points on the outer boundary of the system ($\underline{r} \in \partial D$) and incident directions of flight ($\underline{\Omega} \cdot \underline{n} < 0$, with $\underline{n}$ = outer normal on ∂D):

$$I(\underline{r}, \underline{\Omega}, \nu, t) = I^b(\underline{r}, \underline{\Omega}, \nu, t) \quad . \tag{1.4}$$

The *thermal radiative transfer* (TRT) equations (1.1), (1.3), and (1.4) are mathematically consistent equations for I and T which are linear in I and nonlinear in T. If T_0 is a constant temperature and

$$Q(\underline{r},\nu,t) = 0 \quad , \tag{1.5a}$$

$$I^i(\underline{r},\underline{\Omega},\nu) = B(\nu,T_0) \quad , \tag{1.5b}$$

$$T^i(\underline{r}) = T_0 \quad , \tag{1.5c}$$

$$I^b(\underline{r},\underline{\Omega},\nu) = B(\nu,T_0) \quad , \tag{1.5d}$$

then the TRT equations have the exact *equilibrium solution*:

$$I(\underline{r},\underline{\Omega},\nu,t) = B(\nu,T_0) \quad , \tag{1.6a}$$

$$T(\underline{r},t) = T_0 \quad . \tag{1.6b}$$

The frequency-dependent TRT equations can be simplified if the opacity is independent of frequency: $\sigma(\nu,t) = \sigma(T)$. In this case, if we define the *grey* (frequency-integrated) intensity and source:

$$\overline{I}(\underline{r},\underline{\Omega},t) = \int_0^\infty I(\underline{r},\underline{\Omega},\nu,t)\,d\nu \quad , \tag{1.7a}$$

$$\overline{Q}(\underline{r},t) = \int_0^\infty Q(\underline{r},\nu,t)\,d\nu \quad , \tag{1.7b}$$

we may integrate Eq. (1.1a) over ν and use Eq. (1.2b) to obtain the following equations for $\overline{I}$ and T:

$$\frac{1}{c}\frac{\partial\overline{I}}{\partial t} + \underline{\Omega}\cdot\underline{\nabla}\overline{I} = \sigma\left(\frac{acT^4}{4\pi} - \overline{I}\right) + \frac{\overline{Q}}{4\pi} \quad , \tag{1.8a}$$

$$c_v\frac{\partial T}{\partial t} = \sigma\left(acT^4 - \int_{4\pi}\overline{I}\,d\Omega'\right) \quad . \tag{1.8b}$$

The coefficients $\sigma(T)$ and $c_v(T)$ in these equations – and, of course, the emission source acT^4 – are nonlinear functions of T. Integration of Eq. (1.3a) over ν yields initial conditions for $\overline{I}$ and T:

$$\overline{I}(\underline{r},\underline{\Omega},0) = \overline{I}^i(\underline{r},\underline{\Omega}) \equiv \int_0^\infty I^i(\underline{r},\underline{\Omega},\nu,t)\,d\nu \quad , \tag{1.9a}$$

$$T(\underline{r},0) = T^i(\underline{r}) \quad , \tag{1.9b}$$

and integration of Eq. (1.4) over ν yields the boundary condition for $\overline{I}$:

$$\overline{I}(\underline{r},\underline{\Omega},t) = \overline{I}^b(\underline{r},\underline{\Omega},t) \equiv \int_0^\infty I^b(\underline{r},\underline{\Omega},\nu,t)\,d\nu \quad , \quad \underline{r}\in\partial D \quad , \quad \underline{\Omega}\cdot\underline{n} < 0 \quad . \tag{1.10}$$

The *grey thermal radiative transfer* (GTRT) equations (1.8)-(1.10) are mathematically consistent equations for $\overline{I}$ and T, which are linear in I but nonlinear in T. If T_0 is a positive constant temperature and

$$\overline{Q}(\underline{r},t) = 0 \quad , \tag{1.11a}$$

$$I^i(\underline{r},\underline{\Omega}) = \frac{acT_0^4}{4\pi} \quad , \tag{1.11b}$$

$$T^i(\underline{r}) = T_0 \quad , \tag{1.11c}$$

$$I^b(\underline{r},\underline{\Omega}) = \frac{acT_0^4}{4\pi} \quad , \tag{1.11d}$$

then the GTRT equations have the exact *equilibrium solution*:

$$I(\underline{r}, \underline{\Omega}, t) = \frac{acT_0^4}{4\pi} \ , \tag{1.12a}$$

$$T(\underline{r}, t) = T_0 \ . \tag{1.12b}$$

In Section 2 of this paper we derive, for both the frequency-dependent and grey equations of thermal radiative transfer, a *linearized* thermal radiative transfer theory, which is accurate for problems in which I and T have small deviations from the equilibrium solutions described above. These linear equations hold for any nonlinear dependence of the opacity and heat capacity on T.

The earlier "Su-Olson" linear model of thermal radiation transport [7] requires an opacity that is independent of ν (the *grey* case) and independent of T, and a heat capacity that varies as T^3. These two assumptions – in particular the second – are rarely satisfied in practice. Nevertheless, we show that the linearized grey thermal radiative transfer equations (derived below) have the same mathematical form as the Su-Olson model, and we discuss the implications of this result.

In Section 3 we present the results of numerical simulations, which demonstrate that for *near-equilib-rium* problems, the linearized grey thermal radiative transfer equations accurately describe the fully nonlinear grey radiative process. Unfortunately, frequency-dependent numerical results are not available at this time.

We conclude this paper with a discussion in Section 4.

2. LINEARIZED (NEAR EQUILIBRIUM) THEORY

To derive the *linearized thermal radiative transfer* (LTRT) equations, we consider frequency-dependent problems in which the source and the initial and boundary conditions for I and T satisfy:

$$Q(\underline{r}, \nu, t) = \varepsilon Q_1(\underline{r}, \nu, t) \ , \tag{2.1a}$$

$$I^i(\underline{r}, \underline{\Omega}, \nu) = B(\nu, T_0) + \varepsilon I_1^i(\underline{r}, \underline{\Omega}, \nu) \ , \tag{2.1b}$$

$$T^i(\underline{r}) = T_0 + \varepsilon T_1^i(\underline{r}) \ , \tag{2.1c}$$

$$I^b(\underline{r}, \underline{\Omega}, \nu, t) = B(\nu, T_0) + \varepsilon I_1^b(\underline{r}, \underline{\Omega}, \nu, t) \ . \tag{2.1d}$$

Here T_0 is any positive constant temperature, ε is a dimensionless parameter, and I_1^i, T_1^i, and I_1^b are specified $O(1)$ functions. If $\varepsilon = 0$, the solution of the resulting thermal radiative transfer problem is the equilibrium solution specified by Eqs. (1.6). If $\varepsilon \ll 1$, then the problem is *near-equilibrium*, and it is intuitive to seek solutions of Eqs. (1.1), (1.3), and (1.4) of the form

$$I(\underline{r}, \underline{\Omega}, \nu, t) = B(\nu, T_0) + \varepsilon I_1(\underline{r}, \underline{\Omega}, \nu, t) + O(\varepsilon^2) \ , \tag{2.2a}$$

$$T(\underline{r}, t) = T_0 + \varepsilon T_1(\underline{r}, t) + O(\varepsilon^2) \ , \tag{2.2b}$$

where I_1 and T_1 are $O(1)$.

Introducing Eqs. (2.2) into Eqs. (1.1) and defining the following quantities (which are independent of T_1):

$$\sigma_0(\nu) \equiv \sigma(\nu, T_0) \quad , \tag{2.3a}$$
$$B_0(\nu) \equiv B(\nu, T_0) \quad , \tag{2.3b}$$
$$B_0^{'}(\nu) \equiv \frac{\partial B}{\partial T}(\nu, T_0) \quad , \tag{2.3c}$$
$$c_{v0} = c_v(T_0) \quad , \tag{2.3d}$$

we readily obtain

$$\frac{1}{c}\frac{\partial}{\partial t}\varepsilon I_1 + \underline{\Omega} \cdot \underline{\nabla} \varepsilon I_1 = [\sigma_0 + O(\varepsilon)] \left[\left(B_0 + B_0^{'} \varepsilon T_1 \right) - (B_0 + \varepsilon I_1) \right] + \varepsilon Q_1 + O(\varepsilon^2) \quad , \tag{2.4a}$$

and

$$[c_{v0} + O(\varepsilon)] \frac{\partial}{\partial t} \varepsilon T_1 = \int_0^\infty \int_{4\pi} [\sigma_0 + O(\varepsilon)] \left[(B_0 + \varepsilon I_1) - \left(B_0 + B_0^{'} \varepsilon T_1 \right) \right] d\Omega' d\nu' + O(\varepsilon^2) \quad . \tag{2.4b}$$

The $O(1)$ terms in these equations automatically cancel out, and we discard the $O(\varepsilon^2)$ terms. The resulting $O(\varepsilon)$ terms directly provide linear equations for I_1 and T_1:

$$\frac{1}{c}\frac{\partial I_1}{\partial t} + \underline{\Omega} \cdot \underline{\nabla} I_1 = \sigma_0 \left(B_0^{'} T_1 - I_1 \right) + Q_1 \quad , \tag{2.5a}$$

$$c_{v,0} \frac{\partial T_1}{\partial t} = \int_0^\infty \int_{4\pi} \sigma_0 \left(I_1 - B_0^{'} T_1 \right) d\Omega' d\nu' \quad . \tag{2.5b}$$

Exact initial conditions specifying I_1 and T_1 at $t = 0$ are obtained by introducing Eqs. (2.1) and (2.2) into Eqs. (1.3):

$$I_1(\underline{r}, \underline{\Omega}, \nu, 0) = I_1^i(\underline{r}, \underline{\Omega}, \nu) \quad , \tag{2.6a}$$
$$T_1(\underline{r}, 0) = T_1^i(\underline{r}) \quad . \tag{2.6b}$$

Also, an exact boundary condition specifying I_1 for $\underline{r} \in \partial D$ and $\underline{\Omega} \cdot \underline{n} < 0$ is obtained by introducing Eqs. (2.1) and (2.2) into Eq. (1.4):

$$I_1(\underline{r}, \underline{\Omega}, \nu, t) = I_1^b(\underline{r}, \underline{\Omega}, \nu, t) \quad . \tag{2.7}$$

The *linearized thermal radiative transfer* (LTRT) equations (2.5)-(2.7) are mathematically consistent linear equations for I_1 and T_1. The solutions I_1 and T_1 are not the specific intensity and material temperature, but rather, the *deviations* of the specific intensity and material temperature from an equilibrium solution; see Eqs. (2.2). Although the source $Q_1 \geq 0$, the functions I_1^i, T_1^i, and I_1^b represent deviations from equilibrium and need not be positive. Hence, the solutions I_1 and T_1 need not be positive. However, if I_1^i, T_1^i, I_1^b, and Q_1 are nonnegative, with at least one of these quantities being positive, then it is straightforward to show that I_1 and T_1 are positive. This result is physically intuitive.

The same linearization can be applied to the grey radiative transfer equations (1.8)-(1.10) for problems in which $\sigma(\nu, T) = \sigma(T)$. Alternatively, one can introduce

$$\sigma_0 = \sigma(\nu, T_0) = \sigma(T_0) \tag{2.8}$$

into the already-linearized TRT equations (2.5)-(2.7) and integrate over ν; the same results are obtained either way. Using the above notation for frequency-dependent problems, we define:

$$\overline{I}_1(\underline{r},\underline{\Omega},t) \equiv \int_0^\infty I_1(\underline{r},\underline{\Omega},\nu,t)\, d\nu \quad , \tag{2.9a}$$

$$\overline{Q}_1(\underline{r},t) \equiv \int_0^\infty Q_1(\underline{r},\nu,t)\, d\nu \quad , \tag{2.9b}$$

and use:

$$\int_0^\infty \frac{\partial B_0}{\partial T} d\nu = \left[\frac{d}{dT}\int_0^\infty B(\nu,T)\, d\nu\right]_{T=T_0} = \left[\frac{d}{dT}\frac{acT^4}{4\pi}\right]_{T=T_0} = \frac{ac}{\pi}T_0^3 \quad . \tag{2.10}$$

Then, we obtain the following linearized grey equations for $\overline{I}_1$ and T_1:

$$\frac{1}{c}\frac{\partial \overline{I}_1}{\partial t} + \underline{\Omega}\cdot\underline{\nabla}\overline{I}_1 = \sigma_0\left(\frac{ac}{\pi}T_0^3 T_1 - \overline{I}_1\right) + \overline{Q}_1 \quad , \tag{2.11a}$$

$$c_{v,0}\frac{\partial T_1}{\partial t} = \sigma_0 \int_{4\pi}\left(\overline{I}_1 - \frac{ac}{\pi}T_0^3 T_1\right) d\Omega' \quad , \tag{2.11b}$$

with initial conditions:

$$\overline{I}_1(\underline{r},\underline{\Omega},0) = \overline{I}_1^i(\underline{r},\underline{\Omega}) \equiv \int_0^\infty I_1^i(\underline{r},\underline{\Omega},\nu)\, d\nu \quad , \tag{2.12a}$$

$$T_1(\underline{r},0) = T_1^i(\underline{r}) \quad , \tag{2.12b}$$

and boundary condition:

$$\overline{I}_1(\underline{r},\underline{\Omega},t) = \overline{I}_1^b(\underline{r},\underline{\Omega}) \equiv \int_0^\infty I_1^b(\underline{r},\underline{\Omega},\nu)\, d\nu \quad , \quad \underline{r}\in\partial D, \quad \underline{\Omega}\cdot\underline{n} < 0 \quad . \tag{2.13}$$

The *linearized grey thermal radiative transfer* (LGTRT) equations (2.11)-(2.13) are mathematically consistent linear equations for $\overline{I}_1$ and T_1. As in the frequency-dependent case, $\overline{I}_1$ and T_1 need not be positive unless the source, initial conditions, and boundary condition are nonnegative, and at least one of these is positive.

The LTRT and LGTRT equations can be rewritten in an equivalent form that is more amenable to physical interpretation and Monte Carlo simulation. Let us define the *equilibrium radiation energy density*

$$U(\underline{r},t) \equiv aT^4(\underline{r},t) = \frac{1}{c}\int_0^\infty \int_{4\pi} B(\nu,T) d\Omega d\nu \quad . \tag{2.14}$$

Then,

$$\begin{aligned} U &= a(T_0 + \varepsilon T_1)^4 \\ &= aT_0^4 + \varepsilon(4aT_0^3)T_1 + O(\varepsilon^2) \\ &= U_0 + \varepsilon U_1 + O(\varepsilon^2) \quad . \end{aligned} \tag{2.15}$$

Thus U_1, the first-order deviation of U from equilibrium, is linearly related to T_1 by

$$U_1(\underline{r},t) = 4aT_0^3 T_1(\underline{r},t) \quad . \tag{2.16}$$

Also, we define the *frequency spectrum*

$$\chi_0(\nu) \equiv \frac{\pi}{acT_0^3}\frac{\partial B}{\partial T}(\nu, T_0) = \frac{\pi}{acT_0^3}B_0'(\nu) \ \ , \tag{2.17}$$

which by Eq. (2.10) satisfies

$$\int_0^\infty \chi_0(\nu)\, d\nu = 1 \ \ . \tag{2.18}$$

Using Eqs. (2.16) and (2.17) to eliminate T_1 and B_0' from Eqs. (2.5)-(2.7), we obtain the linear equations:

$$\frac{1}{c}\frac{\partial I_1}{\partial t} + \underline{\Omega}\cdot\underline{\nabla} I_1 + \sigma_0 I_1 = \sigma_0 \frac{\chi_0 c}{4\pi} U_1 + Q_1 \ \ , \tag{2.19a}$$

$$\left(\frac{c_{v,0}}{4aT_0^3}\right)\frac{\partial U_1}{\partial t} + \left(\int_0^\infty \chi_0\sigma_0 d\nu'\right) cU_1 = \int_0^\infty \int_{4\pi} \sigma_0 I_1 d\Omega' d\nu' \ \ , \tag{2.19b}$$

with initial conditions

$$I_1(\underline{r}, \underline{\Omega}, \nu, 0) = I_1^i(\underline{r}, \underline{\Omega}, \nu) \ \ , \tag{2.20a}$$

$$U_1(\underline{r}, 0) = U_1^i(\underline{r}) \equiv 4aT_0^3 T_1^i(\underline{r}) \ \ , \tag{2.20b}$$

and boundary condition:

$$I_1(\underline{r}, \underline{\Omega}, \nu, t) = I_1^b(\underline{r}, \underline{\Omega}, \nu, t) \ \ , \ \ \underline{r}\in\partial D \ \ , \ \ \underline{\Omega}\cdot\underline{n} < 0 \ \ . \tag{2.21}$$

If $\sigma_0(\nu) = \sigma_0 =$ constant, then integration of Eqs. (2.19)-(2.21) over ν yields the following equivalent form of the LGTRT equations:

$$\frac{1}{c}\frac{\partial \overline{I}_1}{\partial t} + \underline{\Omega}\cdot\underline{\nabla}\overline{I}_1 + \sigma_0 \overline{I}_1 = \frac{\sigma_0 c}{4\pi} U_1 + \overline{Q}_1 \ \ , \tag{2.22a}$$

$$\left(\frac{c_{v,0}}{4aT_0^3}\right)\frac{\partial U_1}{\partial t} + \sigma_0 c U_1 = \sigma_0 \int_{4\pi} \overline{I}_1 d\Omega' \ \ , \tag{2.22b}$$

with initial conditions

$$\overline{I}_1(\underline{r}, \underline{\Omega}, 0) = \overline{I}_1^i(\underline{r}, \underline{\Omega}) \ \ , \tag{2.23a}$$

$$U_1(\underline{r}, 0) = U_1^i(\underline{r}) \ \ , \tag{2.23b}$$

and boundary condition

$$\overline{I}_1(\underline{r}, \underline{\Omega}, t) = \overline{I}_1^b(\underline{r}, \underline{\Omega}, t) \ \ , \ \ \underline{r}\in\partial D \ \ , \ \ \underline{\Omega}\cdot\underline{n} < 0 \ \ . \tag{2.24}$$

The LTRT equations (2.19)-(2.21) [or the LGTRT equations (2.22)-(2.24)] have the advantage that the two unknowns I_1/c and U_1 [or $\overline{I}_1/c$ and U_1] are energy densities; this makes a particle transport (and hence Monte Carlo) interpretation of these equations more straightforward.

In fact, the above LGTRT equations (2.22)-(2.24) have already been proposed and implemented as a linear model for grey thermal radiation transport. Su and Olson [7] have shown that if σ is independent of ν and T, and c_v is proportional to T^3, then linear grey radiation transport equations are obtained that have exactly the same mathematical form as Eqs. (2.22)-(2.24). However, the assumptions of Su and Olson on the temperature dependence of σ and c_v are not met in practice, so until now it was not clear whether this model accurately describes any physical radiative transfer process. However, the above analysis shows that the "Su-Olson" equations in fact describe the deviations from equilibrium of *all* grey near-equilibrium thermal radiative transfer problems.

To our knowledge, the frequency-dependent LTRT equations (2.19)-(2.21) are new and have not been previously proposed in the literature.

3. NUMERICAL RESULTS

Now, by comparing the output of two Monte Carlo test codes – one solving the nonlinear GTRT equations with the Fleck and Cummings Implicit Monte Carlo technique [1], the other solving the LGTRT equations using an analog technique developed recently by Ahrens and Larsen [8] – we demonstrate that the linearized radiative transfer equations are an accurate description of the nonlinear radiative transfer process for near-equilibrium problems.

We consider three 1-D grey problems that represent deviations from equilibrium. The physical system is a 3.0 cm slab, with opacity $\sigma = 10/T^3$ cm^{-1} and constant heat capacity $c_v = 0.1$ jk/(cm^3-keV), initially at equilibrium with temperature $T_0 = 0.5$ keV. (Note: 1 jk = 1 jerk = 10^9 Joules.) The right boundary of the slab is reflecting. At $t = 0$, a surface source with temperature T_b is imposed on the left boundary of the slab. In the first problem, $T_b =$ 0.525 keV (representing a 5% jump in temperature). In the second problem, $T_b =$ 0.55 keV (a 10% jump in temperature), and in the third problem, $T_b =$ 0.6 keV (a 20% jump in temperature). Because the radiation energy is proportional to T^4, the first problem represents a 22% jump in energy, the second represents a 46% jump in energy, and the third represents a 107% jump in energy.

For the nonlinear Fleck-Cummings method, we employed a 0.01 cm spatial grid and a time step of 0.01 sh. (Note: 1 sh = 1 shake = 10^{-8} sec.) This space-time grid is sufficient to fully resolve the solution. (The analog Ahrens-Larsen method does not require a spatial grid or time step, except for tallies.) All simulations employed a surface source with 10^8 particles/jk. In Figures 1 through 3, we display the temperature for the three problems at $t = 1$ and 4 sh.

In these three problems, a "warm" thermal wave at temperature T_b propagates into the left edge of a slab, which is initially in equilibrium at a cooler temperature T_0. The linearized theory requires the intensity and temperature deviations from equilibrium to be small, of $O(\varepsilon)$, and it calculates these deviations with $O(\varepsilon^2)$ error. Thus, the linearized equations should have the smallest relative errors in the first problem (5% temperature deviation from equilibrium), larger relative errors in the second problem (10% deviation from equilibrium), and even larger relative errors in the third problem (20% deviation from equilibrium).

The results shown in Figures 1-3 demonstrate this trend. The linearized theory is most accurate when the temperature deviations from equilibrium are an the order of 5% or less, and due to the nonlinearities in the physical problem, this approximate theory becomes less accurate as the deviation from equilibrium increases.

To obtain sufficient statistical accuracy that small deviations were calculated accurately, it was necessary for us to run all simulations with a large number of particles. To have generated accurate solutions for

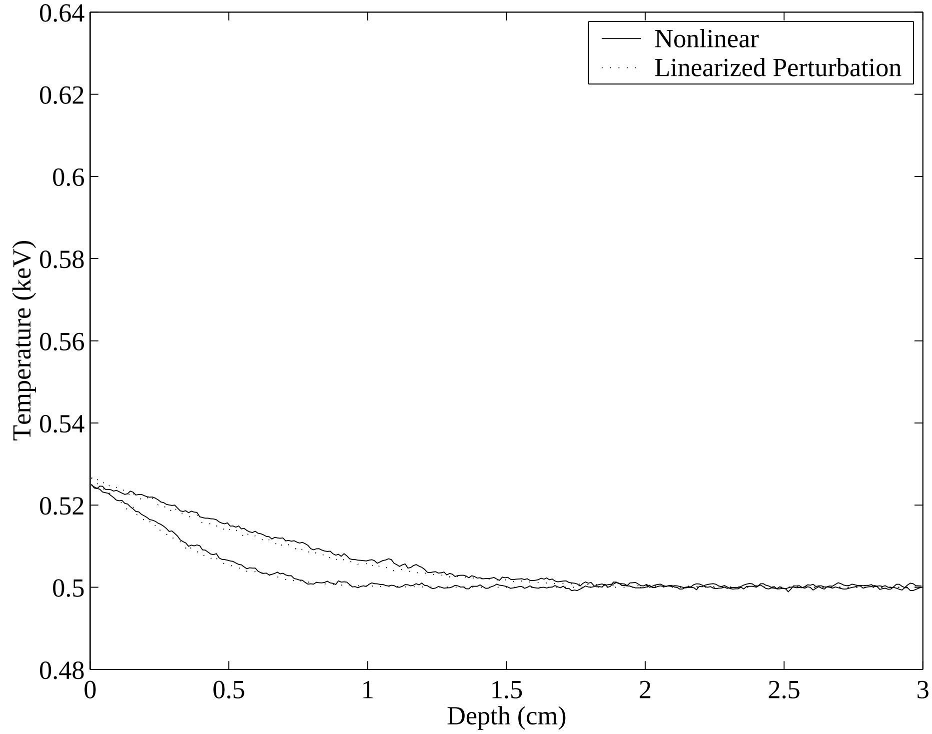

Figure 1. T_b=0.525 keV, T_0 = 0.5 keV (5% ΔT, 22% ΔE)

problems with smaller than 5% temperature deviations, or with significantly smaller statistical errors, would have been beyond the capacity of our computing resources.

4. DISCUSSION

In this paper, we have presented a new linearized theory for near-equilibrium thermal radiative transfer problems. We have also demonstrated that for such problems, accurate numerical solutions of the (nonlinear) GTRT equations and the (linearized) LGTRT equations agree closely.

This work has theoretical and practical consequences. On the theoretical level, it is of interest that nonlinear thermal radiative transfer problems possess physically realistic linear regimes, where theoretical results may be developed that may not exist for more general nonlinear regimes. On the practical level, it is notoriously difficult to obtain benchmark solutions of nonlinear thermal radiative transfer problems – whereas linear problems, although still not easy, can be solved numerically with much greater confidence. For example, the work of Su and Olson [7] was motivated by the desire to generate benchmark solutions for linear problems, against which fully nonlinear algorithms could be tested. The analog Monte Carlo method for the Su-Olson equations recently developed by Ahrens and Larsen [8] enables one to solve LGTRT problems with only statistical errors. In principle, benchmark-quality solutions of arbitrary-geometry 3-D linear grey problems can be obtained using this method. Solutions with smaller truncation errors than the Fleck-Cummings method (for sufficiently small Δx and Δt) can also be obtained using the Carter-Forest method [2].

In addition, if linearized (near-equilibrium) problems possess an analog Monte Carlo solution method, then

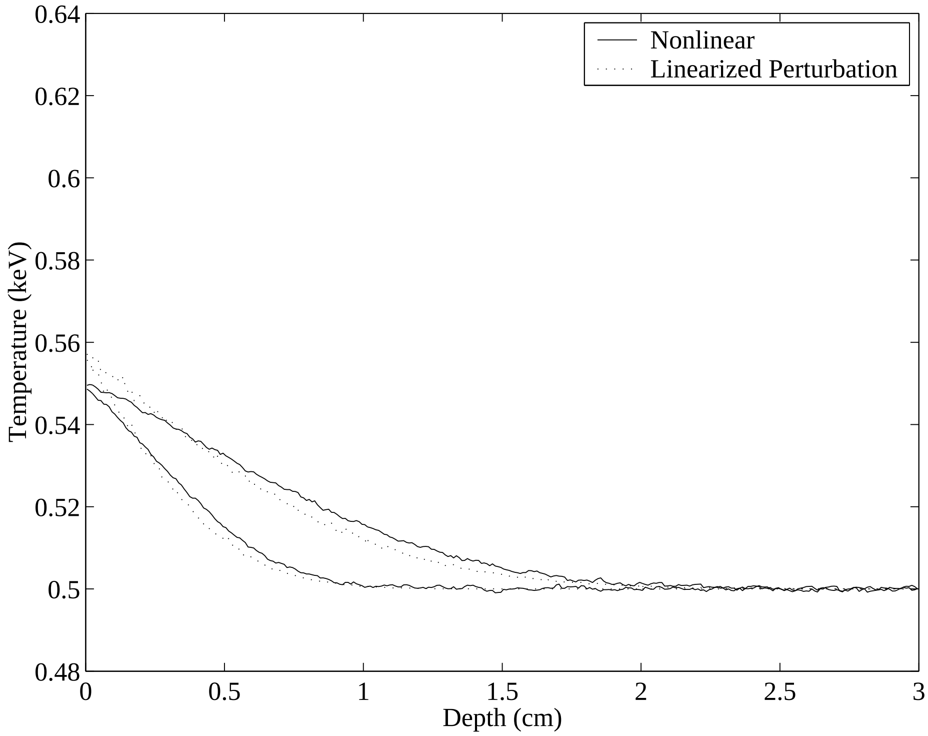

Figure 2. T_b=0.55 keV, T_0 = 0.5 keV (10% ΔT, 46% ΔE)

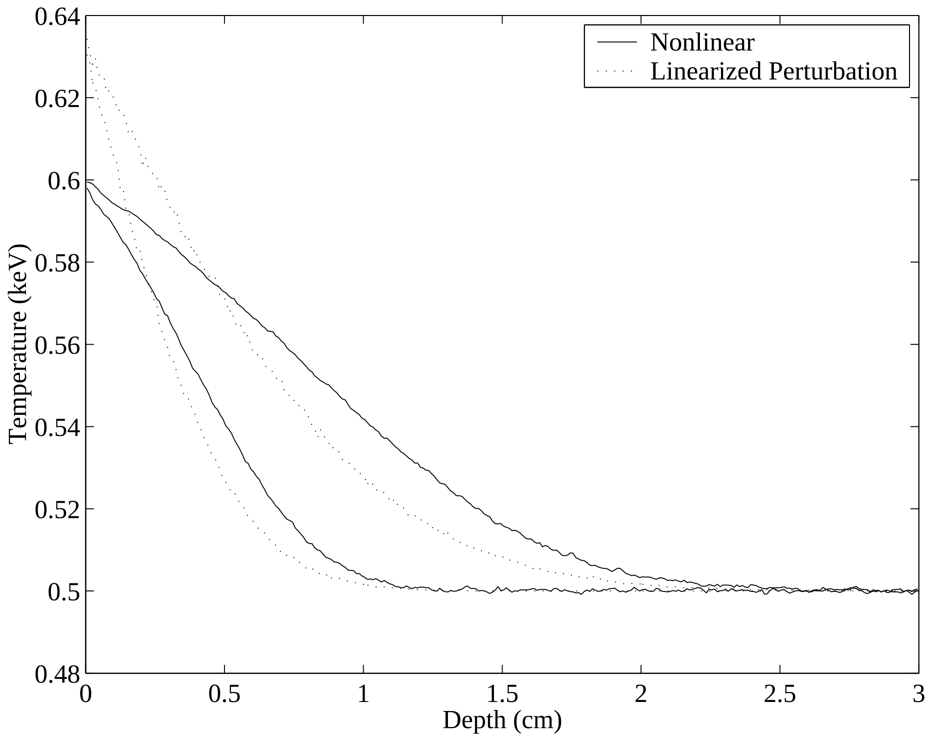

Figure 3. T_b=0.6 keV, T_0 = 0.5 keV (20% ΔT, 107% ΔE)

it becomes a sensible challenge to develop an approximate Monte Carlo technique for nonlinear problems that reduces to the analog method for linear near-equilibrium problems. For example, it is quite easy to show that the widely-used IMC technique of Fleck and Cummings [1] does *not* reduce to an analog Monte Carlo method for near-equilibrium problems. However, the method of Carter and Forest [2] reduces to an analog simulation within each time step, and the method of Ahrens and Larsen [8] becomes completely analog for linear problems

Efforts such as these, together with the development of accurate calculational techniques for the frequency-dependent LTRT equations and comparisons with fully-nonlinear TRT algorithms, are projects that must await future work.

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