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SEQUENTIAL PROBABILITY RATIO TEST ROBUSTNESS WITH RESPECT TO SERIALLY-CORRELATED SIGNALS

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ABSTRACT

In this paper, empirical evidence is presented on the robustness of the Sequential Probability Ratio Test (SPRT). The test utilizes user-specified false and missed alarm probabilities and detects the statistical changes in process signals at the earliest possible time. Earlier research has shown that, even when the analyzed signal distribution is not Gaussian, but a normal distribution is used in the theoretical analysis, SPRT appears to be robust. However, the user-specified false alarm probability may not be met when the analyzed signals contain serial-correlation. The Stochastic Parameter Simulation System has been used to obtain signals with various degrees of serial-correlation. These signals were analyzed by SPRT and it was demonstrated that the number of SPRT false alarms increases as the degree of serial-correlation increases. It was also found that SPRT is robust in analyzing data with a small degree of serial-correlation and a quantifiable "false alarm robustness threshold" was determined. Although this conclusion is based on the analysis of a relatively small number of observations, we believe that it is widely applicable and may have a powerful impact on signal monitoring and validation techniques that use SPRT as a fault detection algorithm.

Key Words: Statistical Tests, Signal Processing, Nuclear Power Plant Applications, Sensor Estimation and Validation Applications

1. INTRODUCTION

In this paper, empirical evidence is presented on the Sequential Probability Ratio Test (SPRT) robustness with respect to the analysis of steady-state, full power nuclear reactor sensor signals. The main objective of the analysis was to determine the SPRT robustness when the test is used to analyze signals with various degrees of serial-correlation.

The SPRT is a statistical hypothesis test that differs from a standard fixed sample test in the way in which statistical observations of the data are employed. In a fixed sample test, a given number of observations are used to select one hypothesis from two or more alternatives. SPRT examines one observation at a time in a data sequence and at some point makes a decision and selects a hypothesis [1].

The SPRT technique provides the basis for detecting the statistical changes in the monitored signals at the earliest possible time and thus provides usable information on the type and location of the disturbance in the signals. The test also provides a dramatic improvement in sensitivity and reliability over the conventional statistical tests. Instead of threshold limits, SPRT utilizes user-specified false-alarm and missed-alarm probabilities, allowing the user to control the likelihood of missed or false alarms. SPRT provides a superior surveillance tool because it is sensitive not only to disturbances in signal mean, but also to very subtle changes in the statistical quality (variance, skewness, bias) of the monitored signal. For sudden, gross failures of sensors or system components the SPRT will annunciate the disturbance just as quickly as a conventional threshold limit check. However, for slow degradation that evolves over a long time period [2], SPRT can annunciate the incipience or onset of a disturbance long before it is apparent to visual inspection and well before conventional threshold limit checks will be tripped.

In this research, SPRT was used in the fault detection module of the Multivariate State Estimation Technique (MSET) [3]. MSET is a pattern recognition tool developed by Argonne National Laboratory (ANL) for system sensor monitoring and fault detection. MSET comprises two important modules: the estimation and the fault detection module. The construction of the MSET model is performed in the **training phase** by determining a system state model that is representative of the normal operation of system sensors. In the **monitoring phase**, each signal is estimated using a nonlinear operator based on the observations of all other signals in the system and the representative system state model. Finally, the difference between the actual observation and the MSET estimation (the MSET residual) is sent to the fault detection module in which SPRT is used to detect sensor output disturbances through the examination of the MSET residuals. SPRT was chosen to perform this task since it is considered to be the most sensitive, accurate and practical method available for real-time applications [1], [3], [4].

2. THEORY

The basic approach taken by the SPRT technique is to analyze successive observations of a discrete physical process by examining the stochastic components of the signal generated from an actual and an estimated value of a sensor that monitors the process. For our particular case, let $\boldsymbol{Y_n}$ represent the discretized difference sample between the real signal and the estimated one at a given moment, $\boldsymbol{t_n}$, in time. If the system is operating normally, it is assumed that the sequence of values $\boldsymbol{\{Y_n\}}$ is normally distributed with mean 0 and variance $\boldsymbol{\sigma^2}$ ($\boldsymbol{H_0}$ Hypothesis). The SPRT is used to test a change in the mean or variance of the sequence.

The SPRT technique, derived from the theory of Wald [1], operates as follows: at each step in a calculation, a test index is calculated and compared to two threshold limits $\boldsymbol{A}$ and $\boldsymbol{B}$ defined below. The mean SPRT examines the mean of the residual signal and the variance SPRT inspects the variance of the residual signal.

For the positive mean SPRT, the problem is to decide between two hypotheses: H_1, where the difference set forms a Gaussian probability density function (pdf) with mean M and variance σ^2; or H_0. The SPRT technique provides a quantitative framework that permits a decision to be made between these two hypotheses with specified misidentification probabilities. If the SPRT accepts H_1, then the analyzed signal is declared degraded. The indicator that helps to decide between the two hypotheses for the positive mean SPRT is:

$$SPRT_{mean}^{pos} = \left[-\frac{1}{2\sigma^2} \sum_{k=1}^{n} M(M - 2y_k) \right] = \frac{M}{\sigma^2} \sum_{k=1}^{n} \left(y_k - \frac{M}{2} \right), \quad (1)$$

For the negative mean SPRT, the problem is again to decide between two hypotheses: H_2, where the difference set forms a Gaussian probability density function with mean $-M$ and variance σ^2; or H_0. The SPRT index for the negative mean test is:

$$SPRT_{mean}^{neg} = \frac{M}{\sigma^2} \sum_{k=1}^{n} \left(-y_k - \frac{M}{2} \right). \quad (2)$$

The variance SPRT for normal distributions examines the variance of the sequence, the system being declared degraded if the sequence exhibits a change in variance by a factor of V or $1/V$, where V, the pre-assigned system disturbance magnitude for the variance test, is a positive number. Again, two tests are performed: nominal variance test where the algorithm decides between H_0 and H_3 (i.e. mean 0 and variance $V\sigma^2$) and inverse variance test where H_0 and H_4 (i.e. mean 0 and variance $(1/V)\sigma^2$) are tested. The associated SPRT indexes for the variance SPRT test are:

$$SPRT_{\text{var}}^{nom} = \frac{1}{2\sigma^2} \left(\frac{V-1}{V} \right) \sum_{k=1}^{n} y_k^2 - \frac{n}{2} \ln V \text{ and} \quad (3)$$

$$SPRT_{\text{var}}^{inv} = \frac{1}{2\sigma^2} (1-V) \sum_{k-1}^{n} y_k^2 + \frac{n}{2} \ln V \, . \quad (4)$$

The threshold limits are related to the misidentification probabilities by the following expressions [1], [3]:

$$A = \frac{\beta}{1-\alpha} \text{ and } \mathrm{B} = \frac{1\text{-}\beta}{\alpha}, \quad (5)$$

where α is the probability of accepting H_1, H_2, H_3 or H_4, respectively, when H_0 is true (i.e. the user-defined false alarm probability) and β is the probability of accepting H_0 when H_1, H_2, H_3 or H_4 are true, respectively (i.e. the user-defined missed alarm probability).

The SPRT module coded in MSET performs both mean and variance tests on a MSET residual signal. To initialize the module for residual signal analysis, the user specifies the system disturbance magnitude for the tests (***M*** and ***V***), the user-defined false alarm probability, α, and the user-defined missed alarm probability, β. The system disturbance magnitude ***M*** for the mean tests specifies the number of standard deviations the residual distribution must shift in the positive or negative direction to trigger an alarm. The system disturbance magnitude ***V*** for the variance tests specifies the fractional change of the residual variance necessary to trigger an alarm. ANL experts recommend numbers ranging from 2 to 4 for both ***M*** and ***V*** [3].

At the beginning of the monitoring phase, all four SPRT indices are set to 0. Then, during each time step of the calculation, the SPRT indices are updated using eq. (1), (2), (3) and (4). Each SPRT index is then compared to the upper (***ln B***) and lower (***ln A***) threshold limits. There are three possible outcomes:

- the SPRT indices are smaller than the lower limit, in which case the process is declared healthy, the test statistic is reset and the sampling continues;
- if any of the SPRT indices are outside the upper limit, an alarm flag corresponding with that particular case is raised, indicating a sensor fault, the test statistic is reset and the sampling continues;
- neither limit has been reached, in which case no decision concerning the process can yet be made and the sampling continues.

MSET is an excellent tool for surveillance and fault detection. However, the steady-state signals that are analyzed by MSET may contain a high degree of serial-correlation. While it is true that the initial serial-correlation is reduced by analyzing the difference between the observation and MSET estimation [5], some serial-correlation may be passed to the MSET residuals that are analyzed by SPRT. As shown previously, theoretically, SPRT analyses data assumed to have a normal distribution. Earlier research at ANL has demonstrated that even if the residual distribution is not Gaussian (but close to being normally distributed) and the normal distribution is used in the calculations, these calculations could still often be used as an accurate estimate of the SPRT properties. In other words, even when the residuals are not Gaussian, SPRT appear to be robust [6]. However, when the MSET residuals contain serially correlated components, the SPRT user-defined false and missed alarm probabilities may not be met since SPRT may be "tricked" into taking a false decision by the "peaks and valleys" introduced by the serially-correlated data [6].

One objective of this research is to show that if SPRT analyzes white and normally distributed MSET residuals for sensors that are **operating normally**, the theoretical criteria and assumptions for SPRT are satisfied and therefore, the user-defined false alarm rates **must** be met. The other objective is to determine the SPRT sensitivity when it is used to analyze serially-correlated MSET residuals.

A Stochastic Parameter Simulation System (SPSS) described in reference [5] is used to completely eliminate or reduce the serial correlation from the MSET residuals. The SPSS uses wavelet and filter banks theory to decompose and perfectly reconstruct a signal ***S***, in the following form [7]:

$$S_t = A_J + \sum_{i=1}^{J} D_i \qquad (6)$$

where $\boldsymbol{A_J}$ is the **wavelet approximation** at the user-defined level of decomposition $\boldsymbol{J}$ and $\boldsymbol{D_i}$ are the signal details. The approximation and the details are completely determined by the wavelet approximation and detail coefficients respectively, obtained in the decomposition filter bank.
A noise model can also be defined, in which a signal $\boldsymbol{S}$ can be seen as [8]:

$$S_t = f_t + \sigma e_t \qquad (7)$$

where $\boldsymbol{f_t}$ is the de-noised signal, $\boldsymbol{\sigma}$ is the noise level (assumed constant in this research) and $\boldsymbol{e_t}$ is the standard white and normally distributed noise with mean 0 and variance 1.

Using the threshold theory on the detail coefficients obtained in the decomposition filter bank, **the wavelet-based de-noised signal**, $\boldsymbol{f_t}$, can be expressed as [8]:

$$f_t = A_J + \sum_{i=1}^{J} D_i' \qquad (8)$$

where the $\boldsymbol{D_i'}$ are the new details completely determined by the threshold coefficients.

One of the SPSS fundamental properties is to find a wavelet-based approximation that, when subtracted from the real signal (in this case, the MSET residuals), the new residuals thus obtained are white and normally distributed. The wavelet-based de-noised signal defined in eq. (8) and a SPSS optimization function are used for this purpose. Another SPSS program option is to use the wavelet approximation, given by eq. (6) (with all details set to 0), to approximate the MSET residuals. It was expected that the simpler wavelet approximation would not completely eliminate the serial-correlation from the MSET residuals, producing therefore additional data points for studying the SPRT robustness when the test is applied to data with different degree of serial-correlation.

In this research, two statistical tests were used to determine the normality (Pearson's χ^2 test [9]) and the whiteness (Bartlett-Durbin procedure [10], [11], [12]) of the analyzed data. The SPSS χ^2 test reports a normality indicator which if it is negative the data is assumed to be normally distributed and if it is positive, the data is assumed to have a non-normal pdf [5]. The SPSS whiteness test reports a normalized Kolmogorov – Smirnov (K-S) indicator which if it smaller than 1, the data is assumed to be white and if it is greater than 1, the data is assumed to contain serial-correlation [5].

3. RESEARCH PROCEDURE

A data file containing 1440 observations of the real output values of 17 sensors recorded at full-power, normal steady-state operation of a nuclear power plant was used for this research. This data was chosen because the observations contained a high degree of serial-correlation. Here, we consider the (artificial) case where the data used to construct the MSET model in the training phase is also input into the MSET estimation module during the monitoring phase and the MSET residuals so generated are analyzed using SPRT. If SPRT is run on the MSET residuals with a user-defined false alarm probability of 0.001, one expects to get one, or at the most two false alarms for the 1440 observations of the "undisturbed signals". Any number of alarms greater than two can be regarded as an excessive false alarm, since the MSET residuals in this case are being produced by the same data set that was used to generate the MSET model and therefore, in the context of the model, must themselves be undisturbed. Therefore, SPRT was run on these MSET residuals and the number of SPRT alarms was recorded.

Then, the SPSS program was used to find two wavelet-based approximations that would eliminate or reduce the serial-correlation from these MSET residuals. A level 8^{th} of decomposition and the Biorthogonal wavelet family was used to find the first wavelet approximation, W1. Second, the SPSS optimization procedure was used to find hundreds of wavelet-based de-noised approximations which, when subtracted from each original MSET residuals, produced signals that were white and normally distributed. One of these wavelet-based de-noised signals, W2, produced by the SPSS optimization procedure using a level 8^{th} of decomposition, Biorthogonal wavelet family and an unscaled noise model, was chosen to illustrate the research results [5].

Both wavelet-based approximations were subtracted from the MSET residuals. The number of alarms given by the SPRT runs on these three sets of residuals (MSET residuals, W1 MSET residuals, obtained by subtracting the wavelet approximation from the MSET residuals and the W2 MSET residuals, obtained by subtracting the wavelet-based de-noised signal from the MSET residuals, respectively) were recorded. For all of the SPRT runs, $\boldsymbol{M}$ and $\boldsymbol{V}$ were set to 4, α was set to 0.001, and β was set to 0.01. Each SPRT run appropriately considered the different variances of the analyzed signals in eq. (1), through (4). The results are presented in the next section.

4. RESULTS

The residuals' normality and whiteness indicators were first analyzed. The results are shown in Figure 1. The figure's abscissa is the normalized K-S (whiteness) indicator; there are four bars for each analyzed signal on the figure's ordinate. The bars show, from left to right, the K-S indicators for the analyzed signal, the MSET residuals, and the W1 and the W2 MSET residuals, respectively. The residuals with non-normal distributions are indicated by a black dot on top of the respective bar. As expected, Figure 1 shows that, even though the worst possible MSET model was used to approximate signals with a high degree of serial-correlation, in most cases

MSET greatly reduced but did not eliminate the serial-correlation in the data. The only exception is signal 6, for which the K-S indicator for the MSET residual increased when compared with the same indicator of the original signal. So, in 16 out of 17 cases, the MSET residuals contained a lower degree of serial correlation than the original data. In 10 out of 17 cases, the MSET residuals are not normally distributed according to the Pearson's χ^2 tests. When SPRT is run on this data, several excessive false alarms are to be expected.

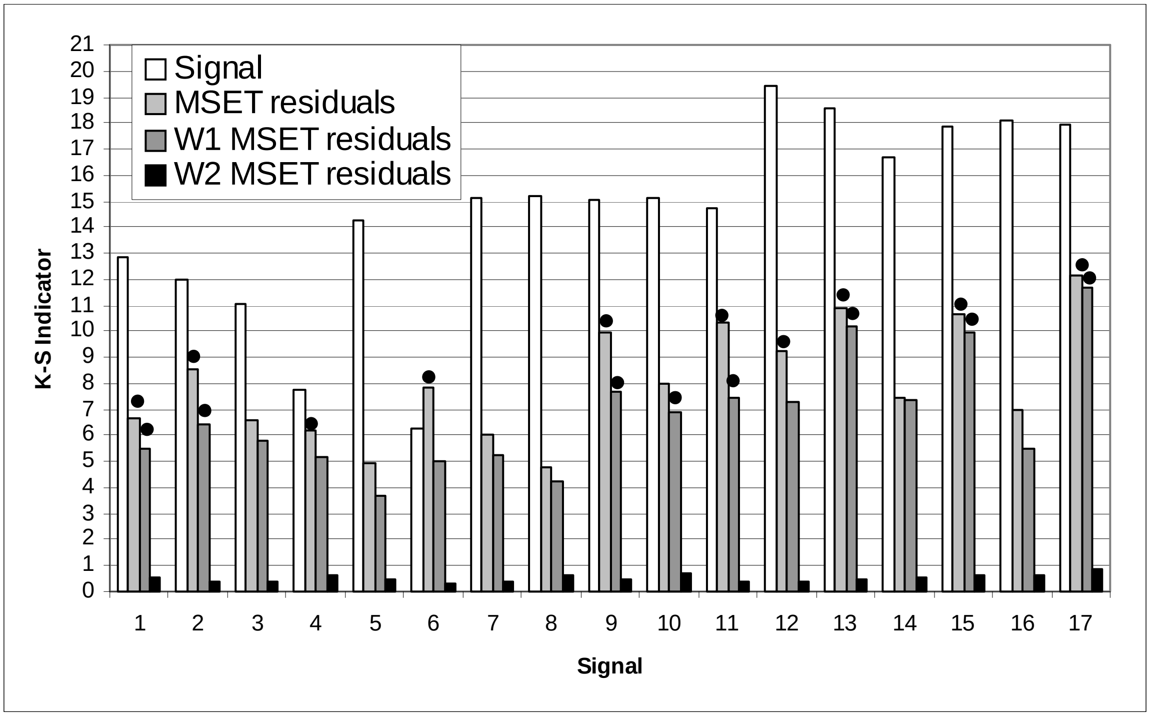

Figure 1 – Whiteness and normality comparison for the original signal, MSET residuals, W1 MSET residuals and W2 MSET residuals.

Figure 1 also shows that the MSET residual serial-correlation for all of the analyzed signals is further reduced, but not eliminated, when the W1 wavelet approximation defined in the previous paragraphs is used to create the W1 MSET residuals. In 8 out of 17 cases the new set of residuals are not normally distributed. We expect to have fewer SPRT excessive false alarms for this data set than for the MSET residual case.

The only wavelet decomposition that completely eliminates the serial-correlation from all MSET residuals is the one that approximates the residuals with the wavelet-based de-noised signal. For this decomposition, the new residuals are also normally distributed. In this case, SPRT is analyzing white and normally distributed residuals and there is no reason to expect that the user-defined false alarm rate (1, maximum 2 alarms for this data set) will not be met.

The SPRT alarm results are summarized in the bar graph presented in Figure 2. The bold horizontal line shows the threshold that distinguishes between excessive false alarms and the expected number of false alarms established by the user-defined false alarm rate, α. The threshold is set to 2 alarms for this data set. For the MSET residuals, the user-defined false alarm rate is met only for two signals (8 and 14). Since a part of the MSET residual serial-correlation is eliminated in the W1 MSET residuals, the user-defined false alarm rate is met for four signals (5, 6, 7 and 8). Apparently, analyzing a residual with a lower degree of serial-correlation usually produces a lower number of SPRT false alarms. There are two exceptions to this observation: signals 14 and 17. In both these cases however, the number of false alarms are comparable.

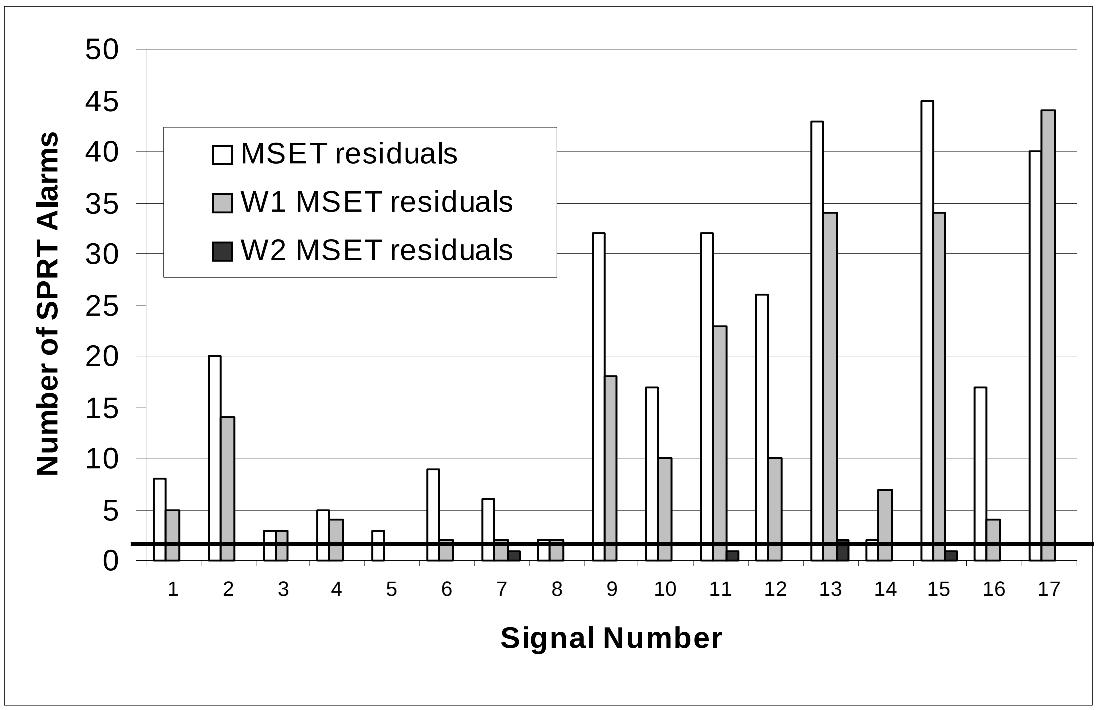

Figure 2 – Comparison of SPRT alarms for the three Sets of MSET residuals. The bold horizontal line is the maximum expected number of false alarms for the data set based on a user-defined false alarm probability rate of 10^{-3}.

As expected, the user-defined false alarm rates are met for all the cases in which SPRT analyzed a white and normally distributed signal (W2 MSET residuals).

Earlier research at ANL showed that, even if the residual distribution is not Gaussian, but the normal distribution is used in theoretical calculations, SPRT appears to be robust [6]. Here,

Figure 2 also shows that the number of SPRT false alarms usually decreases when the degree of serial-correlation is reduced in the residuals. This fact suggests that SPRT may be robust even when it analyzes residuals with a small degree of serial-correlation. To investigate SPRT robustness, the normalized K-S indicator reflecting the degree of serial-correlation in the data was plotted against the number of SPRT false alarms. The results are shown in Figure 3. The data is classified into two groups in Figure 3: the first group represents non-normally distributed residuals and the other group, the normally distributed residuals.

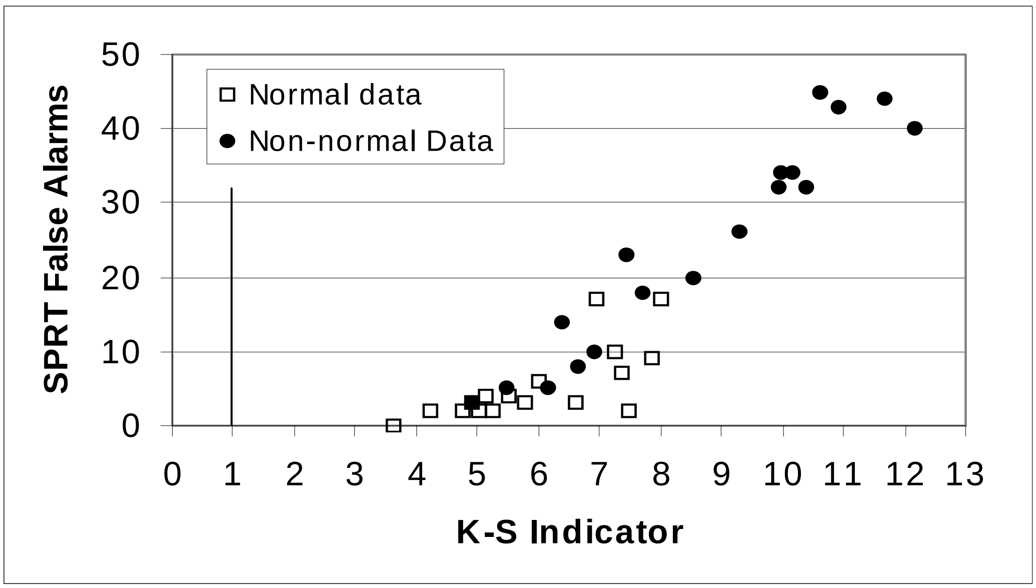

Figure 3 – SPRT alarms vs. K-S indicator for normally and non-normally distributed residuals. A K-S indicator for serially–correlated data with a threshold of 3.5 is suggested for SPRT robustness.

Three very important conclusions can be drawn from Figure 3. First, the higher the degree of serial-correlation in the residuals, the larger is the number of SPRT false alarms. This conclusion was also reported in earlier ANL research [6]. Second, for the same normalized K-S indicator, (i.e., for the same degree of serial-correlation in the residuals), SPRT usually gives more false alarms when it is used to analyze non-normally distributed residuals than for normally distributed residuals. This is in perfect accordance with the SPRT theoretical characteristics, since a normal distribution is assumed in SPRT calculations.

Finally, Figure 3 suggests a threshold value of about 3.5 for the normalized K-S indicator (that represents the degree of serial-correlation in the data in this research), under which no SPRT excessive false alarms are recorded. In other words, **for a small degree of serial-correlation in the residuals, SPRT also appears to be robust**. This conclusion is new and may have a powerful impact for systems like MSET that use SPRT for sensor validation. This means that, even if the MSET residuals are not normally distributed and contain a small degree of serial-correlation, SPRT user-defined false alarm rates may be met. In this case, there is no need to further reduce or eliminate the serial-correlation in the MSET residuals.

5. CONCLUSIONS

This research demonstrates the potential benefits of integrating a methodology similar to SPSS into signal monitoring methods like MSET, which uses SPRT as fault detection algorithm. When MSET residuals contain serially-correlated components, the SPRT user-defined false alarm rates are not usually met. The research demonstrated that by using SPSS to normalize the MSET residuals and eliminate the residual serial-correlation, the user-defined SPRT false alarm rate can be met.

However, further research is needed to completely integrate SPSS with MSET. The research has to focus on the modality of eliminating the serial-correlation from the MSET residuals in the **monitoring phase**. This study shows that such an endeavor may be worthwhile.

For the analyzed data, it was also demonstrated that the number of SPRT false alarms increases as the degree of serial-correlation in data increases. Therefore, the robustness of SPRT with respect with the degree of the serial-correlation in the analyzed data was also studied. It was found that SPRT is robust in analyzing data with a small degree of serial-correlation. The SPRT "false alarm robustness threshold" appears to be for data with a normalized serial-correlation K-S indicator of less than 3.5. Although this conclusion is empirical and is based on the analysis of a relatively small number of signals and observations, we believe that it may be widely applicable and may have a powerful impact on signal monitoring and validation methods that use SPRT as a fault detection algorithm.

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