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THIRD ORDER NODAL FORMULATION RELATING ASSEMBLY POWER AND REACTIVITY

Gustavo Alonso[1], Arturo Delfin
Instituto Nacional de Investigaciones Nucleares
Km 36.5 Carretera Mexico-Toluca, Ocoyoacac, Estado de Mexico 52245, Mexico
galonso@nuclear.inin.mx, adl@nuclear.inin.mx

Edmundo del Valle
Instituto Politecnico Nacional
Escuela Superior de Física y Matemáticas
Unidad Profesional "Adolfo Lopez Mateos", Col. Lindavista 07738, D. F., Mexico
edmundo@esfm.ipn.mx

ABSTRACT

The linear reactivity model is a very fast an efficient way to perform multicyle fuel management analysis, however in the first model proposed by Driscoll requires a fine tuning to predict results comparable to those predicted by 3 dimensional analysis. In the modified scheme proposed by Park it uses an empirical formulation to account for the reactivity leakage penalty. These two previous models use a finite difference technique based on a nodal formulation. In both of these models there are still problems in the adequate prediction of both average core burnup and individual batch burnup prediction. In this work the five-block mesh-centered finite difference schemes based also on a nodal formulation are used as a way to obtain more accurate predictions of local peaking factor, individual batch burnup and average core burnup.

Keywords: Linear Reactivity Model, Fuel Management, Multicycle Analysis

1. INTRODUCTION

In Driscoll's book [1] details are given for a nodal formulation based on a classical finite difference technique relating power and reactivity. The starting points are the diffusion equations corresponding to the steady state 1½ energy groups. This model is lately written as a single equation in terms of the fast neutron flux and an equivalent infinite medium reactivity that is calculated using homogenized two-group constants for each fuel assembly. The final result of this is a very simple expression relating power and reactivity by means of an iterative technique.

Following only this ideas some crucial parameters are clearly identified; the so called power sharing factor (θ) and the leakage reactivity (ρ_L). The first one depends on *h*, the size of the fuel assembly and M^2, the migration area, while the second one is directly related to albedo and this to the diffusion coefficient *D*, *h*, and M^2.

[1] Also at Instituto Politécnico Nacional

To avoid the dependence of q and r_L on such constants, Driscoll et al. [1] suggested to use values empirically determined by a trial and error fine tuning comparing the results given by the code RPM (Reload Power Mapping) and those from QUANDRY, a two-group analytical nodal code [2]. Their results for a beginning-of-life, beginning-of-cycle power map for the Salem-1 PWR using RPM and QUANDRY show that errors are in the range 1% to 10%.

Park et al. [3] have developed a modified model, which uses nodal power coupling relations to calculate batch power fraction directly instead of using fine tuning to calculate q. However this model still uses an empirical formula to consider the leakage reactivity penalty, where some tuning is necessary to get predictions comparable to those obtained from 3-dimensional analysis. The results presented by Park et al. show a maximum error of 2% for whole core burnup calculation. The cores used in the problems presented by Park use just one type of assembly decreasing the power sharing effect given in actual cores and it does not show a power peaking map to know the range of the error in power peaking prediction.

The main problem in the use of the linear reactivity model is the adequate batch burnup prediction as can be seen in the paper written by Alonso and Parish [4] where the agreement between core average burnup is less than 5% meanwhile the difference between batch burnup prediction can be up to 30%. The Linear Reactivity Model used by Alonso and Parish is exactly the one given by Driscoll.

In both the Driscoll and Park approaches, a nodal formulation based on a finite difference technique was used. On the other hand, Hennart and del Valle [5] developed general five-block mesh-centered finite difference schemes from RT*k* (Raviart-Thomas of index k, $k\geq 0$) nodal schemes to solve numerically the one-group diffusion equation for a given source. Part of this work is based on the ideas there described.

Thus, the preceding sections are organized as follow: in Section 2 a description of the diffusion model is given. In Section 3 we described the algorithm to solve the one-group diffusion equation for a given source using a five-block mesh-centered finite difference scheme. Then, we will start Section 4 with the simplest case, the RT0 nodal scheme, just to show that it provides the same equations obtained in [1,3]. Then, in Section 5, the RT1 nodal scheme will be introduced describing the details to get the discretized equations in distinct situations. In Section 6 numerical results are given for a model problem and finally some conclusions are offered.

2. THE BASICS OF THE DRISCOLL'S METHOD

It is well known that the two-group diffusion equations can be written as follows.

$$-D_1 \nabla^2 f_1 + S_{R1} f_1 = \frac{1}{k}\left(nS_{f1} f_1 + nS_{f2} f_2\right) \tag{1a}$$

$$-D_2 \nabla^2 f_2 + S_{R2} f_2 = S_{s_{1\to 2}} f_1 \tag{1b}$$

Where ϕ_1 and ϕ_2 are the fast and thermal neutron flux and the physical constants have the conventional meaning. Now, considering that thermal leakage is negligible then thermal flux can be written in terms of the fast flux as follows

$$\phi_2 = \frac{\Sigma_{s_{1\to2}}}{\Sigma_{R2}}\ \phi_1 \tag{2}$$

Substituting Eq. (2) in (1a) one gets the so-called 1½ groups diffusion approximation given by

$$-D_1\nabla^2\phi_1 + \Sigma_{R1}\phi_1 = \frac{1}{k}\left(\nu\Sigma_{f1} + \frac{\nu\Sigma_{f2}\Sigma_{s_{1\to2}}}{\Sigma_{R2}}\right)\phi_1 \tag{3}$$

Now, introducing the migration area given by

$$M^2 = \frac{D_1}{\Sigma_{R1}} \tag{4}$$

and the reactivity equivalent to an homogenized infinite medium

$$\rho = 1 - \frac{\Sigma_{R1}}{\nu\Sigma_{f1} + \dfrac{\nu\Sigma_{f2}\Sigma_{s_{1\to2}}}{\Sigma_{R2}}} \tag{5}$$

one gets finally

$$\nabla^2\phi_1 + \frac{\rho}{1-\rho}\frac{1}{M^2}\phi_1 = 0 \tag{6}$$

where the local power density q''' and ϕ_1 are related by

$$\phi_1 = q'''\left(\frac{\nu}{\kappa}\frac{M^2}{D_1}\right)(1-\rho). \tag{7}$$

We will assume that the coefficient $(\nu M^2/\kappa D_1)$ is constant for a given fuel assembly where κ is the fission energy production conversion factor. Doing so, region-average fast flux can be replaced by the product of q''' and 1- ρ .

3. MESH-CENTERED FINITE DIFFERENCES FROM A NODAL FORMULATION

The one group diffusion equation can be written as follows for a given source S

$$-\nabla\cdot D\nabla\phi + \Sigma\phi = S, \forall(x,y)\in\Omega \tag{8}$$

subject to the following boundary conditions

$$\frac{\partial \boldsymbol{f}}{\partial n} = 0 \ , \quad (x,y)\, \boldsymbol{e}\, G_1 \tag{9a}$$

$$\boldsymbol{f} \ = \ 0 \ , \quad (x,y)\, \boldsymbol{e}\, G_2 \tag{9b}$$

Now, for a given polynomial approximation $\boldsymbol{f}_h$

$$\boldsymbol{f}_h \ = \ \sum \boldsymbol{f}_i\, u_i(x,y) \tag{10}$$

the finite element Galerkin approach leads to the following algebraic system

$$\left(K_x + K_y + M\right)\boldsymbol{F} = Q \tag{11}$$

Where the elementary or local matrices corresponding to K_x, K_y and M are given by

$$k^e_{x_{ij}} : \ = D_{i,j}\, \frac{Dy_j}{Dx_i} \int_{-1}^{+1} \int_{-1}^{+1} \frac{\partial u_i}{\partial \boldsymbol{x}}\, \frac{\partial u_j}{\partial \boldsymbol{x}}\, d\boldsymbol{x}\, d\boldsymbol{h} \tag{12a}$$

$$k^e_{y_{ij}} : \ = \ D_{i,j}\, \frac{Dx_i}{Dy_j} \int_{-1}^{+1} \int_{-1}^{+1} \frac{\partial u_i}{d\boldsymbol{h}}\, \frac{\partial u_j}{d\boldsymbol{h}}\, d\boldsymbol{x}\, d\boldsymbol{h} \tag{12b}$$

and

$$m_{ij} : = \frac{\Sigma_{i,j}\, \Delta x_i\, \Delta y_j}{4} \int_{-1}^{+1} \int_{-1}^{+1} u_i\, (\boldsymbol{x},\boldsymbol{h})\, u_j\, (\boldsymbol{x},\boldsymbol{h})\, d\boldsymbol{x}\, d\boldsymbol{h} \tag{13}$$

where the u_i are the local basis functions.

The procedure described in [5] was done for RT*k* nodal finite element schemes. In general, the RT*k* finite nodal scheme interpolates the zero up to *(k+1)* Legendre moments per edge and the (00), ….., up to *(k+1, k+1)* Legendre moments per cell leading to a total of *(k+1)(k+5*) interpolation parameters.

Let us introduce a couple of polynomial spaces that are useful to understand the properties of the RT*k* nodal schemes. First we introduce the polynomial space Q_{kl} defined by

$$Q_{kl} = \left\{ x^a\, y^b \ , 0 \le \boldsymbol{a} \le k \, , 0 \le \boldsymbol{b} \le k \right\} \tag{14}$$

with in particular

$$Q_k \equiv Q_{kk} \tag{15}$$

as well as the polynomial space $\boldsymbol{R}_k$ given by

$$\boldsymbol{R}_k = \left\{ x^{\alpha} y^{\beta} \text{ , } 0 \le \boldsymbol{\alpha} + \boldsymbol{\beta} \le k \right\} \tag{16}$$

Now, over the reference cell $C \equiv [-1,+1]\times[-1,+1]$ corresponding to the cell $C_{ij} \equiv [x_{i-1/2}, x_{i+1/2}]\times[y_{j-1/2}, y_{j+1/2}]$, cell Legendre moments are defined as follow

$$\boldsymbol{\phi}_C^{ij} \equiv \frac{\int_{-1}^{+1}\int_{-1}^{+1} P_{ij}(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{\phi}_h(\boldsymbol{\xi},\boldsymbol{\eta})\, d\boldsymbol{\xi}\, d\boldsymbol{\eta}}{N_i N_j}; \quad i, j = 0,1,\ldots,k \tag{17}$$

where $P_{ij}(\boldsymbol{\xi},\boldsymbol{\eta})=P_i(\boldsymbol{\xi})P_j(\boldsymbol{\eta})$, $P_i(\boldsymbol{\xi})$ being the Legendre polynomial of degree i in $\boldsymbol{\xi}$ and $N_i=2/(2i+1)$. Moreover, edge Legendre moments are given by

$$\boldsymbol{\phi}_E^{i} \equiv \frac{\int_{-1}^{+1} P_i(s_E)\boldsymbol{\phi}_h(\boldsymbol{\xi}_E,\boldsymbol{\eta}_E)\, ds_E}{N_i}; \quad i = 0,1,\ldots,k \tag{18}$$

where E stands by left $(i\text{-}1/2,j)$, right $(i+1/2,j)$, bottom $(i,j\text{-}1/2)$, and top $(i,j+1/2)$ edges respectively; $\boldsymbol{\xi}_E$ or $\boldsymbol{\eta}_E$ is ± 1, depending on the particular edge considered; and the other coordinate along that edge is s_E. The local approximation for each of these nodal schemes is built on a polynomial space S_h of dimension $(k+1)(k+5)$ that is given by $Q_{k+2,k} \cup Q_{k,k+2}$.

Once that a particular nodal scheme is chosen, to evaluate integrals in equations (12) and (13) respectively, Hennart and del Valle [5] apply $(2k+2)$ points open Newton-Cotes to release the coupling between left and right, and also between bottom and top interpolation parameters in the cells and $(k+2)$ points Gauss-Radau quadratures to lump the mass matrices in such a way, that only just a few elements in the diagonal are non-zero.

In Theorem 1 of Reference [5] is shown that the L^2 error norm of the RTk nodal schemes is $O(h^{\lambda+1})$ where $\lambda = min(\ell, m, n+1)$, being ℓ the greatest positive integer such that $\boldsymbol{R}_\ell \subset S_h$, m the number of edge Legendre moments that are kept between adjacent cells, and n is an index related to the accuracy of the quadrature rules used, exact for the polynomial space Q_{2n+1}. This means that the L^2 error norm of the RT0 and RT1 nodal schemes are $O(h^2)$ and $O(h^3)$, respectively.

4. THE SECOND ORDER NODAL FORMULATION

As it is expected the RT0 nodal scheme is the simplest one. In this formulation the 2-points open Newton-Cotes quadrature is applied to evaluate the stiffness matrix elements and the 2-points Gauss-Radau quadrature to evaluate the mass matrix elements to get the following set of equations

$$2D_{i,j}\,\frac{\left(f^{0}_{i+1/2,j}-f^{00}_{i,j}\right)}{\Delta x_i}+2D_{i+1,j}\,\frac{\left(f^{0}_{i+1/2,j}-f^{00}_{i+1,j}\right)}{\Delta x_{i+l}}=0 \tag{19a}$$

$$2D_{i,j}\,\frac{\left(f^{0}_{i,j+1/2}-f^{00}_{i,j}\right)}{\Delta y_j}+2D_{i,j+1}\,\frac{\left(f^{0}_{i,j+1/2}-f^{00}_{i,j+1}\right)}{\Delta y_{j+l}}=0 \tag{19b}$$

and

$$\begin{aligned}&D_{i,j}\,\frac{\Delta y_j}{\Delta x_i}\left(-2f^{0}_{i-1/2,j}-2f^{0}_{i+1/2,j}+4f^{00}_{i,j}\right)+D_{i,j}\,\frac{\Delta x_i}{\Delta y_j}\left(-2f^{0}_{i,j-1/2}-2f^{0}_{i,j+1/2}+4f^{00}_{i,j}\right)\\&+\Sigma_{i,j}\Delta x_i\Delta y_j f^{00}_{i,j}=\Delta x_i\Delta y_j S^{00}_{i,j}\end{aligned} \tag{20}$$

Eqs. (19a) and (19b) can be interpreted as the continuity of the zeroth Legendre moment of neutron current in the x and y directions respectively, introducing the zeroth Legendre moment of J_x at left and right edges of cell *(i,j)*

$$J^{0}_{xR(i,j)}=\frac{2D_{i,j}}{Dx_i}\left(f^{0}_{i+1/2,j}-f^{00}_{i,j}\right)\;;\quad J^{0}_{xL(i,j)}=-\frac{2D_{i,j}}{Dx_i}\left(f^{0}_{i-1/2,j}-f^{00}_{i,j}\right) \tag{21a}$$

and the zeroth Legendre moment of J_y at bottom and top edges of cell *(i,j)*

$$J^{0}_{yT(i,j)}=\frac{2D_{i,j}}{Dy_j}\left(f^{0}_{i,j+1/2}-f^{00}_{i,j}\right)\;;\quad J^{0}_{yB(i,j)}=-\frac{2D_{i,j}}{Dy_j}\left(f^{0}_{i,j-1/2}-f^{00}_{i,j}\right) \tag{21b}$$

These equations are particularly useful when one or two cells are reflectors. For instance, in the case that cell *(i + 1,j)* is a reflector, then by using the albedo definition

$$a=J^{0-}_{xR(i,j)}/J^{0+}_{xR(i,j)}=\frac{\frac{1}{4}f^{0}_{i+1/2,j}-\frac{1}{2}J^{0}_{xR(i,j)}}{\frac{1}{4}f^{0}_{i+1/2,j}+\frac{1}{2}J^{0}_{xR(i,j)}}=\frac{\left(1-\frac{4D_{i,j}}{h}\right)f^{0}_{i+1/2,j}+\frac{4D_{i,j}}{h}f^{00}_{i,j}}{\left(1+\frac{4D_{i,j}}{h}\right)f^{0}_{i+1/2,j}-\frac{4D_{i,j}}{h}f^{00}_{i,j}} \tag{22}$$

one gets an expression relating the $f^{0}_{i+1/2,j}$ and $f^{00}_{i,j}$ given by

$$f^{0}_{i+1/2,j}=d^{0}_{xi,j}\,f^{00}_{i,j}\;;\quad d^{0}_{xi,j}=\left[1+\frac{h(1-a)}{4D_{ij}(1+a)}\right]^{-1} \tag{23a}$$

where $h=\Delta x_i=\Delta y_j$ is the width of the fuel assembly. Analogously, when cell *(i,j+1)* is a reflector one gets

$$f^{0}_{i,j+1/2} = d^{\,0}_{yi,j}\, f^{\,00}_{i,j} \;;\; d^{\,0}_{yi,j} = \left[1 + \frac{h(1-a)}{4D_{ij}(1+a)}\right]^{-1} \tag{23b}$$

Now, solving (19a) for $f^{0}_{i+1/2,j}$ and (19b) for $f^{0}_{i,j+1/2}$ using the diffusion coefficient of Eq. (6)

$$f^{\,0}_{i+1/2\;,j} = \frac{1}{2}\left(f^{\,00}_{i+1\;,j} + f^{\,00}_{i\;,j}\right) \;;\quad f^{\,0}_{i-1/2\;,j} = \frac{1}{2}\left(f^{\,00}_{i\;,j} + f^{\,00}_{i-1\;,j}\right) \tag{24a}$$

$$f^{\,0}_{i\;,j+1/2} = \frac{1}{2}\left(f^{\,00}_{i\;,j+1} + f^{\,00}_{i\;,j}\right) \;;\quad f^{\,0}_{i\;,j-1/2} = \frac{1}{2}\left(f^{\,00}_{i\;,j} + f^{\,00}_{i\;,j-1}\right) \tag{24b}$$

then Eq. (20) becomes

$$\sum_{m} f^{00}_{m} - 4\,f^{00}_{i,j} + \frac{r_{i,j}}{(1-r_{i,j})M^{2}_{i,j}}\, f_{i,j} = 0 \tag{25}$$

and by using

$$f^{\,00}_{i\;,j} \approx \left(1 - r_{\;i\;,j}\right) q^{\,00}_{i\;,j} \tag{26}$$

Eq. (25) can be transformed and rearranged to yield

$$q^{00}_{i,j} = \frac{\frac{1}{4}\sum_{m} q^{00}_{m}\left(1-r_{m}\right)}{1-q_{i,j}\, r_{i,j}} \;;\quad \text{or} \qquad f^{00}_{i,j} = \frac{\frac{1}{4}\sum_{m} f^{00}_{m}\left(1-r_{m}\right)}{1-q_{i,j}\, r_{i,j}} \tag{27}$$

where $q_{i,j} = 1 + h^{2}/4M^{2}_{i,j}$ and the sum with index m is carried out over cells *(i-1,j), (i+1,j), (i,j-1),* and *(i,j+1).* Eq. (27) changes when one or two cells are reflectors, namely

$$f^{00}_{i,j} = \frac{\frac{1}{4-R}\sum_{m} f^{00}_{m}\left(1-r_{m}\right)}{1-q_{R}\left(r_{i,j} - r_{LR}\right)} \tag{28}$$

where R is the number of assembly edges facing a reflector, the sum with index m is carried out over the *4-R* fuel assemblies, and q_{R} and r_{LR} are given in Table I in terms of h and M^2, where $g = 2d - 1$. This shows that this method is equivalent to the one described in [1,3].

Table I. Power sharing and reactivity leakage factors

R (Number of reflectors)	q_R	r_{LR}
0	$1+h^2/4M^2$	0
1	$(4q_0-g)/3$	$(1-g)/(4q_0-g)$
2	$2q_0-g$	$(1-g)/(2q_0-g)$

5. THE THIRD ORDER NODAL FORMULATION

This formulation is based on the RT1 nodal scheme, that when 4-points open Newton-Cotes quadrature is applied to evaluate the stiffness matrix elements and the 3-points Gauss-Radau quadrature to evaluate the mass matrix elements one gets the following set of equations

$$\frac{2D_{i,j}}{Dx_i}(3f^0_{i+1/2,j}-3f^{00}_{i,j}-2f^{10}_{i,j})+\frac{2D_{i+1,j}}{Dx_{i+1}}(3f^0_{i+1/2,j}-3f^{00}_{i+1,j}+2f^{10}_{i+1,j})=0 \quad (29a)$$

$$\frac{2D_{i,j}}{\Delta x_i}(f^1_{i+1/2,j}-f^{01}_{i,j}-\frac{2}{3}f^{11}_{i,j})+\frac{2D_{i+1,j}}{\Delta x_{i+1}}(f^1_{i+1/2,j}-f^{01}_{i+1,j}+\frac{2}{3}f^{11}_{i+1,j})=0 \quad (29b)$$

$$\frac{2D_{i,j}}{\Delta y_j}(3f^0_{i,j+1/2}-3f^{00}_{i,j}-2f^{01}_{i,j})+\frac{2D_{i,j+1}}{\Delta y_{j+1}}(3f^0_{i,j+1/2}-3f^{00}_{i,j+1}+2f^{01}_{i,j+1})=0 \quad (30a)$$

$$\frac{2D_{i,j}}{\Delta y_j}(f^1_{i,j+1/2}-f^{10}_{i,j}-\frac{2}{3}f^{11}_{i,j})+\frac{2D_{i,j+1}}{\Delta y_{j+1}}(f^1_{i,j+1/2}-f^{10}_{i,j+1}+\frac{2}{3}f^{11}_{i,j+1})=0 \quad (30b)$$

and

$$D_{i,j}\frac{\Delta y_j}{\Delta x_i}(-6f^0_{i-1/2,j}-6f^0_{i+1/2,j}+12f^{00}_{i,j})+D_{i,j}\frac{\Delta x_i}{\Delta y_j}(-6f^0_{i,j-1/2}-6f^0_{i,j+1/2}+12f^{00}_{i,j})+\Sigma\Delta x_i\Delta y_j f^{00}_{i,j}=\Delta x_i\Delta y_j S^{00}_{i,j} \quad (31a)$$

$$D_{i,j}\frac{\Delta y_j}{\Delta x_i}(-2f^1_{i-1/2,j}-2f^1_{i+1/2,j}+4f^{01}_{i,j})+D_{i,j}\frac{\Delta x_i}{\Delta y_j}(4f^0_{i,j-1/2}-4f^0_{i,j+1/2}+8f^{01}_{i,j})+\frac{1}{3}\Sigma\Delta x_i\Delta y_j f^{01}_{i,j}=\frac{1}{3}\Delta x_i\Delta y_j S^{01}_{i,j} \quad (31b)$$

$$D_{i,j}\frac{\Delta y_j}{\Delta x_i}(4f^0_{i-1/2,j}-4f^0_{i+1/2,j}+8f^{10}_{i,j})+D_{i,j}\frac{\Delta x_i}{\Delta y_j}(-2f^0_{i,j-1/2}-2f^0_{i,j+1/2}+4f^{10}_{i,j})+\frac{1}{3}\Sigma\Delta x_i\Delta y_j f^{10}_{i,j}=\frac{1}{3}\Delta x_i\Delta y_j S^{10}_{i,j} \quad (31c)$$

$$D_{i,j}\frac{\Delta y_j}{\Delta x_i}(\frac{4}{3}f^1_{i-1/2,j}-\frac{4}{3}f^1_{i+1/2,j}+\frac{8}{3}f^{11}_{i,j})+D_{i,j}\frac{\Delta x_i}{\Delta y_j}(\frac{4}{3}f^1_{i,j-1/2}-\frac{4}{3}f^0_{i,j+1/2}+\frac{8}{3}f^{11}_{i,j})+\frac{1}{9}\Sigma\Delta x_i\Delta y_j f^{11}_{i,j}=\frac{1}{9}\Delta x_i\Delta y_j S^{11}_{i,j} \quad (31d)$$

This time, Eqs. (29a) and (29b) can be interpreted as the continuity of the zeroth and first Legendre moment of the neutron current in the x-direction defining

$$J^{0}_{xR(i,j)} = +\frac{2D_{i,j}}{\Delta x_i}(3\phi^{0}_{i+1/2,j} - 3\phi^{00}_{ij} - 2\phi^{10}_{i,j});\quad J^{0}_{xL(i,j)} = -\frac{2D_{i+1,j}}{\Delta x_{i+1}}(3\phi^{0}_{i-1/2,j} - 3\phi^{00}_{ij} + 2\phi^{10}_{i,j}) \tag{32a}$$

$$J^{1}_{xR(i,j)} = +\frac{2D_{i,j}}{\Delta x_i}(\phi^{1}_{i+1/2,j} - \phi^{01}_{ij} - \frac{2}{3}\phi^{11}_{i,j});\quad J^{1}_{xL(i,j)} = -\frac{2D_{i+1,j}}{\Delta x_{i+1}}(\phi^{1}_{i-1/2,j} - \phi^{01}_{ij} + \frac{2}{3}\phi^{11}_{i,j}) \tag{32b}$$

and Eqs. (31a)-(31d) are weighted cell balances, (00),(01), (10), and (11) Legendre moments. Analogous expressions can be obtained for the zeroth and first Legendre moments of the neutron current in the y-direction.

Eqs. (29a) and (29b) are particularly useful to relate fast neutron flux at an edge with its Legendre moments in a cell via albedo when a reflector is facing a fuel assembly. For instance if cell *(i+1,j)* is a reflector then from albedo definition one would get the following

$$\alpha = J^{0-}_{xR(i,j)} / J^{0+}_{xR(i,j)} = \frac{\frac{1}{4}\phi^{0}_{i+1/2,j} - \frac{1}{2}J^{0}_{xR(i,j)}}{\frac{1}{4}\phi^{0}_{i+1/2,j} + \frac{1}{2}J^{0}_{xR(i,j)}} \quad \text{and} \quad \alpha = J^{1-}_{xR(i,j)} / J^{1+}_{xR(i,j)} = \frac{\frac{1}{4}\phi^{1}_{i+1/2,j} - \frac{1}{2}J^{1}_{xR(i,j)}}{\frac{1}{4}\phi^{1}_{i+1/2,j} + \frac{1}{2}J^{1}_{xR(i,j)}} \tag{33}$$

Now, substituting Eqs. (32a) and (32b) in (33) and solving for $\phi^{0}_{i+1/2,j}$ and $\phi^{1}_{i+1/2,j}$ one gets

$$\phi^{0}_{i+1/2,j} = \delta^{0}_{xi,j}(\phi^{00}_{i,j} + \frac{2}{3}\phi^{10}_{i,j}),\ \delta^{0}_{xi,j} = \frac{1}{1 + \left(\frac{1-\alpha}{1+\alpha}\right)\frac{h}{12D_{i,j}}} \tag{34a}$$

$$\phi^{1}_{i+1/2,j} = \delta^{1}_{xi,j}(\phi^{01}_{i,j} + \frac{2}{3}\phi^{11}_{i,j}),\ \delta^{1}_{xi,j} = \frac{1}{1 + \left(\frac{1-\alpha}{1+\alpha}\right)\frac{h}{4D_{i,j}}} \tag{34b}$$

Similar expressions can be found for $\phi^{0}_{i,j+1/2}$ and $\phi^{1}_{i,j+1/2}$ if there is a reflector in cell *(i,j+1)*.

Now, applying Eqs. (29a) and (29b) for the Eq. (6) one may obtain the following

$$\phi^{0}_{i+1/2,j} = \frac{1}{2}(\phi^{00}_{i,j} + \phi^{00}_{i+1,j}) + \frac{1}{3}(\phi^{10}_{i,j} - \phi^{10}_{i+1,j}) \quad \text{and} \quad \phi^{0}_{i-1/2,j} = \frac{1}{2}(\phi^{00}_{i-1,j} + \phi^{00}_{i,j}) + \frac{1}{3}(\phi^{10}_{i-1,j} - \phi^{10}_{i,j}) \tag{35}$$

and analogously

$$f^{0}_{i,j+1/2} = \frac{1}{2}(f^{00}_{i,j} + f^{00}_{i,j+1}) + \frac{1}{3}(f^{01}_{i,j} - f^{01}_{i,j+1}) \text{ and } f^{0}_{i,j-1/2} = \frac{1}{2}(f^{00}_{i,j-1} + f^{00}_{i,j}) + \frac{1}{3}(f^{01}_{i,j-1} - f^{01}_{i,j}) \tag{36}$$

Substitution of Eqs. (35) and (36) in Eqs. (31a)-(31d) for the equation of interest with no source term gives a set of equations for the Legendre moments of the fast flux in the cell *(i,j)* . Finally, to obtain expressions for the Legendre moments of power fraction in cell *(i,j)* one substitutes the fast flux by the power in the cell times one minus the reactivity, the final result is then divided by the total power to get the following set of equations

$$\begin{aligned}(1-q^{00} r_{i,j}) f^{00}_{i,j} &= \frac{1}{4}\sum_{m}(1-r_m) f^{00}_m \\ &+ \frac{1}{6}\left[(1-r_{i-1,j}) f^{10}_{i-1,j} - (1-r_{i+1,j}) f^{10}_{i+1,j}\right] + \frac{1}{6}\left[(1-r_{i,j-1}) f^{01}_{i,j-1} - (1-r_{i,j+1}) f^{01}_{i,j+1}\right]\end{aligned} \tag{37a}$$

$$\begin{aligned}(1-q^{01} r_{i,j}) f^{01}_{i,j} &= \frac{3}{22}\sum_{m_x}(1-r_{m_x}) f^{01}_{m_x} - \frac{2}{11}\sum_{m_y}(1-r_{m_y}) f^{01}_{m_y} \\ &+ \frac{3}{11}\left[(1-r_{i,j+1}) f^{00}_{i,j+1} - (1-r_{i,j-1}) f^{00}_{i,j-1}\right] + \frac{1}{11}\left[(1-r_{i-1,j}) f^{11}_{i-1,j} - (1-r_{i+1,j}) f^{01}_{i,j+1}\right]\end{aligned} \tag{37b}$$

$$\begin{aligned}(1-q^{10} r_{i,j}) f^{10}_{i,j} &= \frac{3}{22}\sum_{m_x}(1-r_{m_x}) f^{10}_{m_x} - \frac{2}{11}\sum_{m_y}(1-r_{m_y}) f^{10}_{m_y} \\ &+ \frac{3}{11}\left[(1-r_{i,j+1}) f^{00}_{i,j+1} - (1-r_{i,j-1}) f^{00}_{i,j-1}\right] + \frac{1}{11}\left[(1-r_{i-1,j}) f^{11}_{i-1,j} - (1-r_{i+1,j}) f^{01}_{i,j+1}\right]\end{aligned} \tag{37c}$$

$$\begin{aligned}(1-q^{11} r_{i,j}) f^{11}_{i,j} &= \frac{1}{8}\sum_{m}(1-r_m) f^{11}_m \\ &+ \frac{3}{16}\left[(1-r_{i,j-1}) f^{10}_{i,j-1} - (1-r_{i,j+1}) f^{10}_{i,j+1}\right] + \frac{3}{16}\left[(1-r_{i-1,j}) f^{01}_{i-1,j} - (1-r_{i+1,j}) f^{01}_{i+1,j}\right]\end{aligned} \tag{37d}$$

where the sum with index m is carried out over cells *(i-1,j), (i+1,j), (i,j-1),* and *(i,j+1),* the other one with index m_x is swept over cells *(i-1,j), (i+1,j),* and finally, sum with index m_y is carried out over cells *(i,j-1),* and *(i,j+1)*. On the other hand q parameters are given by

$$q^{00} = 1 + \frac{h^2}{12M^2_{i,j}}, q^{01} = 1 + \frac{h^2}{22M^2_{i,j}}, q^{10} = 1 + \frac{h^2}{22M^2_{i,j}}, \text{ and } q^{11} = 1 + \frac{h^2}{32M^2_{i,j}} \tag{38}$$

Similar expressions may be obtained in those cases where one or two cells corresponds to reflector material. For instance if cell *(i+1,j)* is not a fuel assembly but a reflector then instead of Eqs. (37a)-(37d) one obtains the following

$$\begin{aligned}\left[1-q^{00}_{P}\left(r_{i,j} - r^{00}_{Lx}\right)\right] f^{00}_{i,j} - \frac{1}{6}(1-r_{i,j}) g^{0}_{xi,j} f^{10}_{i,j} &= \frac{1}{4}\sum_{m}(1-r_m) f^{00}_m \\ &+ \frac{1}{6}(1-r_{i-1,j}) f^{10}_{i-1,j} + \frac{1}{6}\left[(1-r_{i,j-1}) f^{01}_{i,j-1} - (1-r_{i,j+1}) f^{01}_{i,j+1}\right]\end{aligned} \tag{39a}$$

$$\left[1-q_P^{01}\left(r_{i,j}-r_{Lx}^{01}\right)\right]f_{i,j}^{01}+\frac{1}{11}(1-r_{i,j1})g_{xi,j}^{1}f_{i,j}^{11}=\frac{3}{22}(1-r_{i-1,j})f_{i-1,j}^{01}-\frac{2}{11}\sum_{m_y}(1-r_{m_y})f_{m_y}^{01}$$
$$+\frac{3}{11}\left[(1-r_{i,j+1})f_{i,j+1}^{00}-(1-r_{i,j-1})f_{i,j-1}^{00}\right]+\frac{1}{11}(1-r_{i-1,j})f_{i-1,j}^{11} \tag{39b}$$

$$\left[1-q_P^{10}\left(r_{i,j}-r_{Lx}^{10}\right)\right]f_{i,j}^{10}+\frac{3}{11}(1-r_{i,j})g_{xi,j}^{0}f_{i,j}^{11}=\frac{3}{22}\sum_{m_x}(1-r_{m_x})f_{m_x}^{10}-\frac{2}{11}(1-r_{i,j-1})f_{i,j-1}^{10}$$
$$-\frac{3}{11}(1-r_{i-1,j})f_{i-1,j}^{00}+\frac{1}{11}\left[(1-r_{i,j-1})f_{i,j-1}^{11}-(1-r_{i,j+1})f_{i,j+1}^{11}\right] \tag{39c}$$

$$\left[1-q_P^{11}\left(r_{i,j}-r_{Lx}^{11}\right)\right]f_{i,j}^{11}-\frac{3}{16}(1-r_{i,j})g_{xi,j}^{1}f_{i,j}^{10}=-\frac{1}{8}\sum_{m}(1-r_{m})f_{m}^{11}$$
$$+\frac{3}{16}\left[(1-r_{i,j-1})f_{i,j-1}^{10}-(1-r_{i,j+1})f_{i,j+1}^{10}\right]-\frac{3}{16}(1-r_{i-1,j})f_{i-1,j}^{01} \tag{39d}$$

where this time sum with index m is carried out over cells *(i-1,j)*, *(i,j-1)*, and *(i,j+1)*, meanwhile

$$q_P^{00}=q^{00}-\frac{g_{xi,j}^{0}}{4};\ r_{Lx}^{00}=-\frac{g_{xi,j}^{0}}{4q_P^{00}};g_{xi,j}^{0}=2d_{xi,j}^{0}-1; \tag{40a}$$

$$q_P^{01}=q^{01}-\frac{2g_{xi,j}^{0}}{11};\ r_{Lx}^{01}=-\frac{2g_{xi,j}^{0}}{11q_P^{01}}; \tag{40b}$$

$$q_P^{10}=q^{10}-\frac{3g_{xi,}^{1}}{22};\ r_{Lx}^{10}=-\frac{3g_{xi,j}^{1}}{22q_P^{10}};g_{xi,j}^{1}=2d_{xi,j}^{1}-1; \tag{40c}$$

$$q_P^{11}=q^{11}-\frac{g_{xi,j}^{1}}{8};\ r_{Lx}^{01}=-\frac{g_{xi,j}^{1}}{8q_P^{11}}; \tag{40d}$$

Finally, if cells *(i+1,j)* and *(i,j+1)* correspond to reflectors then Eqs. (39) are replaced by

$$\left[1-q_P^{00}\left(r_{i,j}-r_{Lxy}^{00}\right)\right]f_{i,j}^{00}-\frac{1}{6}(1-r_{i,j})g_{xi,j}^{0}f_{i,j}^{10}-\frac{1}{6}(1-r_{i,j})g_{yi,j}^{0}f_{i,j}^{01}=\frac{1}{4}\sum_{m}(1-r_{m})f_{m}^{00}$$
$$+\frac{1}{6}(1-r_{i-1,j})f_{i-1,j}^{10}+\frac{1}{6}(1-r_{i,j-1})f_{i,j-1}^{01} \tag{41a}$$

$$\left[1-q_P^{01}\left(r_{i,j}-r_{Lxy}^{01}\right)\right]f_{i,j}^{01}-\frac{3}{11}(1-r_{i,j})g_{xi,j}^{0}f_{i,j}^{00}-\frac{1}{11}(1-r_{i,j1})g_{xi,j}^{1}f_{i,j}^{11}=\frac{3}{22}(1-r_{i-1,j})f_{i-1,j}^{01}$$
$$-\frac{2}{11}(1-r_{i,j-1})f_{i,j-1}^{01}-\frac{3}{11}(1-r_{i,j-1})f_{i,j-1}^{00}+\frac{1}{11}(1-r_{i-1,j})f_{i-1,j}^{11} \tag{41b}$$

$$\left[1-q_{P}^{10}\left(r_{i,j}-r_{Lxy}^{10}\right)\right]f_{i,j}^{10}-\frac{1}{11}(1-r_{i,j})g_{xi,j}^{0}\,f_{i,j}^{00}-\frac{3}{11}(1-r_{i,j})g_{yi,j}^{1}\,f_{i,j}^{11}=\frac{3}{22}(1-r_{i,j-1})\,f_{i,j-1}^{10}$$
$$-\frac{2}{11}(1-r_{i,j-1})\,f_{i,j-1}^{10}-\frac{3}{11}(1-r_{i-1,j})\,f_{i-1,j}^{00}+\frac{1}{11}(1-r_{i,j-1})\,f_{i,j-1}^{11} \qquad (41c)$$

$$\left[1-q_{P}^{11}\left(r_{i,j}-r_{Lxy}^{11}\right)\right]f_{i,j}^{11}-\frac{3}{16}(1-r_{i,j})g_{xi,j}^{1}\,f_{i,j}^{01}-\frac{3}{16}(1-r_{i,j})g_{yi,j}^{1}\,f_{i,j}^{10}=-\frac{1}{8}\sum_{m}(1-r_{m})f_{m}^{11}$$
$$-\frac{3}{16}(1-r_{i,j-1})\,f_{i,j-1}^{10}-\frac{3}{16}(1-r_{i-1,j})\,f_{i-1,j}^{01} \qquad (41d)$$

where this time sum with index m is carried out over cells *(i-1,j)* and *(i,j-1)*, meanwhile all the other parameters involved are given by

$$q_{P}^{00}=q^{00}-\frac{(g_{xi,j}^{0}+g_{yi,j}^{0})}{4};\ r_{Lxy}^{00}=-\frac{(g_{xi,j}^{0}+g_{yi,j}^{0})}{4q_{P}^{00}}; \qquad (42a)$$

$$q_{P}^{01}=q^{01}-\frac{1}{11}\left(2g_{yi,j}^{0}+\frac{3}{2}g_{xi,j}^{1}\right);\ r_{Lxy}^{01}=-\frac{1}{11q_{P}^{01}}\left(2g_{yi,j}^{0}+\frac{3}{2}g_{xi,j}^{1}\right); \qquad (42b)$$

$$q_{P}^{10}=q^{10}-\frac{1}{11}\left(2g_{xi,j}^{0}+\frac{3}{2}g_{yi,j}^{1}\right);\ r_{Lxy}^{10}=-\frac{1}{11q_{P}^{10}}\left(2g_{yi,j}^{0}+\frac{3}{2}g_{xi,j}^{1}\right); \qquad (42c)$$

$$q_{P}^{11}=q^{11}-\frac{(g_{xi,j}^{1}+g_{yi,j}^{1})}{8};\ r_{Lxy}^{01}=-\frac{(g_{xi,j}^{1}+g_{yi,j}^{1})}{8q_{P}^{11}}; \qquad (42d)$$

Once that the set of equations for each cell of the core has been built its numerical solution is obtained following the same iterative technique mentioned in [1] that consists in the calculation of the power fractions in the *(n+1)* iteration once that those corresponding to iteration *(n)* are given, particularly from an initial guess. On each iteration the power fractions are normalized to unity over the whole core. The iterations are stopped when a standard convergence criteria is satisfied.

6. NUMERICAL RESULTS FOR A MODEL PROBLEM

To verify that the third order MCFD scheme here above described provides better results than the second order MCFD, we proposed a homogeneous 2-group model problem. This problem has an analytical solution given by the product of cosine functions in directions x and y. We compute reactivity from the following set of cross sections: D_1=1.0cm, D_2=1.10cm, $S_{a1}=0.05$ cm^{-1}, $S_{a2}=0.05$ cm^{-1}, $n\Sigma_{f1}=0.06$ cm^{-1}, $n\Sigma_{f2}=0.02$ cm^{-1}, and $\Sigma_{s1\rightarrow 2}=0.005$ cm^{-1}. The assembly width is h=25 cm. We considered only the right-up quadrant with a grid of 9x9 assemblies and reflective boundary conditions in the left and bottom edges and zero neutron flux at the right and top edges. The power fractions obtained with the second and third order MCFD schemes were compared with those obtained analytically for this model problem and are shown in Table II. This table shows on each cell the analytical, RT0, and RT1 power fractions, giving also the errors for RT0 and RT1. From this table one realizes that the maximum errors for RT0 and RT1 MCFD schemes are 11.5% and 6.1% respectively meanwhile the minimum errors are 0.3% and 0.05%.

Table II. Numerical results for the test problem

0.214 0.221(3.6) 0.224(5.0)	0.207 0.214(3.2) 0.213(2.6)	0.194 0.200(2.6) 0.197(1.2)	0.176 0.179(1.8) 0.177(0.5)	0.152 0.153(0.8) 0.152(0.5)	0.123 0.123(0.3) 0.124(0.9)	0.091 0.089(1.5) 0.092(1.7)	0.056 0.054(2.9) 0.057(2.9)	0.019 0.018(3.8) 0.019(4.1)
0.635 0.664(4.6) 0.658(3.7)	0.615 0.642(4.3) 0.624(1.4)	0.577 0.599(3.7) 0.577(0.05)	0.522 0.537(2.9) 0.518(0.7)	0.45 0.459(1.9) 0.447(0.7)	0.365 0.368(0.7) 0.364(0.3)	0.269 0.268(0.5) 0.271(0.5)	0.165 0.162(1.9) 0.168(1.6)	0.056 0.054(2.9) 0.057(2.9)
1.036 1.098(6.0) 1.063(2.5)	1.005 1.062(5.7) 1.007(0.2)	0.943 0.990(5.1) 0.932(1.2)	0.852 0.888(4.2) 0.837(1.8)	0.735 0.759(3.2) 0.722(1.9)	0.597 0.609(2.1) 0.588(1.4)	0.44 0.443(0.8) 0.437(0.6)	0.269 0.268(0.5) 0.271(0.5)	0.091 0.089(1.5) 0.092(1.7)
1.406 1.509(7.3) 1.430(1.7)	1.364 1.459(7.0) 1.355(0.6)	1.279 1.361(6.4) 1.254(2.0)	1.156 1.221(5.6) 1.126(2.6)	0.998 1.044(4.5) 0.972(2.7)	0.81 0.837(3.4) 0.792(2.2)	0.597 0.609(2.1) 0.588(1.4)	0.365 0.368(0.7) 0.364(0.3)	0.123 0.123(0.3) 0.124(0.9)
1.734 1.882(8.6) 1.756(1.3)	1.681 1.819(8.2) 1.664(1.0)	1.577 1.697(7.6) 1.539(2.4)	1.426 1.522(6.8) 1.382(3.1)	1.231 1.301(5.7) 1.192(3.1)	0.998 1.044(4.5) 0.972(2.7)	0.735 0.759(3.2) 0.722(1.9)	0.45 0.459(1.9) 0.447(0.7)	0.152 0.153(0.8) 0.152(0.5)
2.008 2.202(9.6) 2.036(1.4)	1.947 2.128(9.3) 1.929(0.9)	1.827 1.985(8.7) 1.784(2.4)	1.651 1.780(7.8) 1.602(3.0)	1.426 1.522(6.8) 1.382(3.1)	1.156 1.221(5.6) 1.126(2.6)	0.852 .888(4.2) .837(1.8)	0.522 0.537(2.9) 0.518(0.7)	0.176 0.179(1.8) 0.177(0.5)
2.222 2.456(10.5) 2.269(2.1)	2.155 2.374(10.2) 2.149(0.2)	2.022 2.214(9.5) 1.988(1.7)	1.827 1.985(8.7) 1.784(2.4)	1.577 1.697(7.6) 1.539(2.4)	1.279 1.361(6.4) 1.254(2.0)	0.943 0.990(5.1) 0.932(1.2)	0.577 0.599(3.7) 0.577(0.05)	0.194 0.200(2.6) 0.197(1.2)
2.368 2.633(11.2) 2.454(3.6)	2.296 2.545(10.8) 2.324(1.2)	2.155 2.374(10.2) 2.149(0.2)	1.947 2.128(9.3) 1.929(0.9)	1.681 1.819(8.2) 1.664(1.0)	1.364 1.459(7.0) 1.355(0.6)	1.005 1.062(5.7) 1.007(0.2)	0.615 0.642(4.3) 0.624(1.4)	0.207 0.214(3.2) 0.213(2.6)
2.442 2.724(11.5) 2.592(6.1)	2.368 2.633(11.2) 2.454(3.6)	2.222 2.456(10.5) 2.269(2.1)	2.008 2.202(9.6) 2.036(1.4)	1.734 1.882(8.6) 1.756(1.3)	1.406 1.509(7.3) 1.430(1.7)	1.036 1.098(6.0) 1.063(2.5)	0.635 0.664(4.6) 0.658(3.7)	0.214 0.221(3.6) 0.224(5.0)

Analytical RT0(error0) RT1(error1)

error$k$ = (Analytical-RTk)/Analytical, k=0,1

CONCLUSIONS

Results for the RT0 scheme are similar to the ones obtained in Ref. [2], it means they are in the error range of 1% to 10% approximately. Furthermore, the use of the scheme RT1 reduces the error to the range of 1% to 6%. It is evident that numerical results show that the RT1 scheme is better than the RT0. Of course this conclusion is valid only for this model problem. It will be necessary to make more tests to ensure the accuracy of the proposed third order MCFD scheme in a general framework.

These results are preliminary and it is still necessary to perform further research on the linear reactivity model to generate a computer program to obtain errors no greater than 5% to predict power fractions, batch burnup and average core burnup in multicycle analysis. Nonetheless we are confident that the third order MCFD scheme can be used instead of the second order one to get better results in the calculation of the power fractions and other parameters of interest.

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