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APPROXIMATIONS TO THE DOMINANCE RATIO USING EFFECTIVE AND INFINITE MULTIPLICATION RESULTS

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ABSTRACT

The dominance ratio for large homogeneous systems that scatter neutrons isotropically can be approximated using the effective multiplication and the infinite multiplication values. The approximation is relatively good even for smaller systems and improves as the system size increases. Use of these approximations also help gauge the effectiveness, or lack thereof, of using reflective boundary conditions in a given geometry. We also test the approximation for a simple multi-group problem and discuss its application to problems with anisotropic scattering.

Key Words: multiplication, dominance ratio, eigenvalue, approximation

1. INTRODUCTION

Several nuclear systems of interest to various nuclear engineering communities are often difficult to model with numerical approximations to the neutron transport equation. Both Monte Carlo and discrete ordinates codes can encounter difficulty in systems that have multiple fission source regions that are loosely coupled – the former because many particle histories are required to obtain good statistics (requires many inactive cycles and results in suspect confidence intervals) and the latter because of ray effects and slow outer iteration convergence. Interest in these loosely-coupled problems has also arisen recently with a benchmarking initiative of the Organization for Economic Co-operation and Development (OECD) [1]. In addition, Monte Carlo codes can encounter difficulty with large systems, as many particles are required to obtain full “communication” throughout the geometry.

In both instances, the numerical evaluation is characterized by a system with a large dominance ratio – the ratio of the fundamental eigenvalue to the first higher mode eigenvalue [2]. One technique that can be used to reduce the dominance ratio is to take advantage of problem symmetries with reflective boundary conditions. Previous work focused on analyzing systems that were loosely coupled by using transport theory Green’s functions and fission matrix analysis [3]. In such systems, we see eigenvalue clusters that result from the individual fission source regions. We now extend the analysis of large dominance ratio systems by examining physically large systems.

In this paper, we use diffusion and transport theory analysis to establish a convenient approximation to the dominance ratio for large systems. Because we are examining large homogeneous systems, diffusion theory is a good approximation to the transport equation. The approximation focuses on using two ubiquitously calculated properties of a nuclear system, the effective multiplication (k_{eff}) and the infinite multiplication (k_∞), to approximate the dominance ratio. We examine several one-group homogeneous geometries in one, two, and three dimensions, provide results that can be applied to these geometries, and discuss some considerations for systems with multi-group energy structures and systems that scatter neutrons anisotropically. These results also have implications with respect to using problem symmetry to reduce the dominance ratio.

2. ONE-DIMENSIONAL ANALYSIS

To begin the analysis, we consider one-dimensional systems that scatter monoenergetic neutrons isotropically. We will consider Cartesian, spherical, and cylindrical systems in turn. The objective is to determine a simple relationship between the dominance ratio and the ratio k_{eff}/k_∞.

2.1. Cartesian Geometry

Using one-group diffusion theory, we find that the eigenvalues (odd and even modes) for a homogeneous slab are given by

$$k_n = \frac{\nu\Sigma_f/\Sigma_a}{1+\frac{D}{\Sigma_a}\left(\frac{n\pi}{\Delta+2\delta}\right)^2} \quad, n = 1, 3, 5, \ldots \quad , \quad k_n = \frac{\nu\Sigma_f/\Sigma_a}{1+\frac{4D}{\Sigma_a}\left(\frac{n\pi}{\Delta+2\delta}\right)^2} \quad , \quad n = 1, 2, 3, \ldots \quad , \tag{1}$$

where D is the diffusion coefficient, Δ is the slab thickness, and δ is the extrapolation distance. The dominance ratio (DR) is given by the ratio of the two largest eigenvalues in Eq. (1), or

$$DR = \frac{1+\frac{D}{\Sigma_a}\left(\frac{\pi}{\Delta+2\delta}\right)^2}{1+\frac{4D}{\Sigma_a}\left(\frac{\pi}{\Delta+2\delta}\right)^2} \quad . \tag{2}$$

If we perform a Taylor expansion of the denominator according to $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$, we see that the dominance ratio is given by

$$DR = 1 - 3\frac{D\pi^2}{\Sigma_a}\frac{1}{(\Delta+2\delta)^2} + 12\left(\frac{D\pi^2}{\Sigma_a}\right)^2\frac{1}{(\Delta+2\delta)^4} - \ldots \quad . \tag{3}$$

If we take the *cube* of the ratio of the fundamental mode to the infinite multiplication factor ($k_\infty = \nu\Sigma_f/\Sigma_a$) and expand into a Taylor series, we find that

$$\left(\frac{k_{eff}}{k_\infty}\right)^3 = 1 - 3\frac{D\pi^2}{\Sigma_a}\frac{1}{(\Delta+2\delta)^2} + 6\left(\frac{D\pi^2}{\Sigma_a}\right)^2\frac{1}{(\Delta+2\delta)^4} - \ldots \quad , \tag{4}$$

which to two terms is identical to the dominance ratio per Eq. (3). In addition, the third term is smaller in Eq. (4) than in Eq. (3) so that the approximation to the dominance ratio will underestimate the actual value. (We see this phenomenon in all of the subsequent analyses, which indicates the possibility of using a correction factor to improve the approximation.) Thus, we see that for a homogeneous slab, we can approximate the dominance ratio by taking the cube of the ratio of the effective multiplication to the infinite multiplication. If we use a reflective boundary condition at the center of the slab, which eliminates all odd eigenmodes, and perform the above analysis, we see that the dominance ratio is now approximated by $(k_{eff}/k_\infty)^8$.

We now employ the Green's Function Method (GFM) [3] as a confirmation of the diffusion theory results. We show the dominance ratio and the approximation derived from diffusion theory, $(k_{eff}/k_\infty)^3$, in Figure 1 as calculated with the GFM using the cross sections shown in Table I. As expected, the approximation to the dominance ratio improves as the slab thickness increases.

Table I. Sample one-dimensional slab cross sections.

Cross Section	Value
Σ_t	1
$\nu\Sigma_f$	0.25
Σ_s	0.9

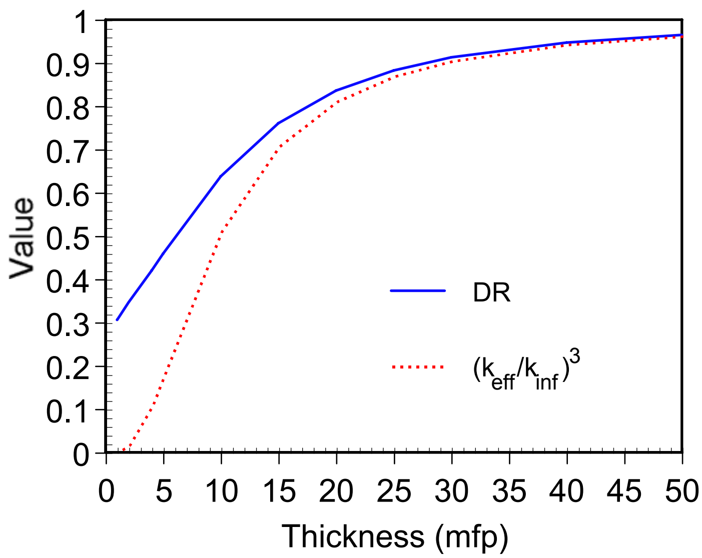

Figure 1. GFM approximation to the dominance ratio for a 1-D slab.

Overall, analytic diffusion theory [Eq. (2)] provides an excellent approximation to the transport dominance ratio. If the dominance ratio is a quantity of interest in support of a general transport calculation, diffusion theory provides an excellent approximation relative to transport theory, as shown in Figure 2. Even for very thin slabs, the relative error between diffusion theory and transport theory is within 10%.

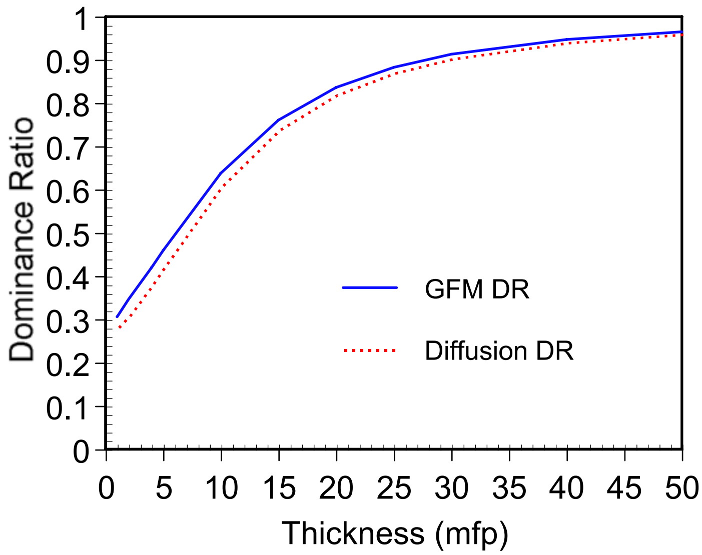

Figure 2. Comparison of 1-D slab dominance ratio for the GFM and diffusion theory.

Examination of the Taylor series expansions in the preceding equations indicates that the approximations to the dominance ratio are a function of the cross sections used. If we assume that $D = 1/3\Sigma_s$, we can show that the analytic diffusion theory approximations are best when $c_s = 0.5$, where $c_s = \Sigma_s/\Sigma_t$. As c_s decreases toward 0.5, the relative errors of the approximation to the dominance ratio for a given slab width also decrease but as c_s decreases below 0.5, the relative errors increase symmetrically. Thus, the relative error for a $c_s = 0.9$ slab are identical to those for a $c_s = 0.1$ slab. We do not see this behavior with the transport analysis. We show the results for slabs of various thickness with $\nu\Sigma_f = 0.25$ and $\Sigma_t = 1$ for three values of Σ_s in Figure 3. As the slab becomes more absorbing, the approximation to the dominance ratio becomes better. As expected, when a material becomes highly absorbing, the diffusion approximation breaks down.

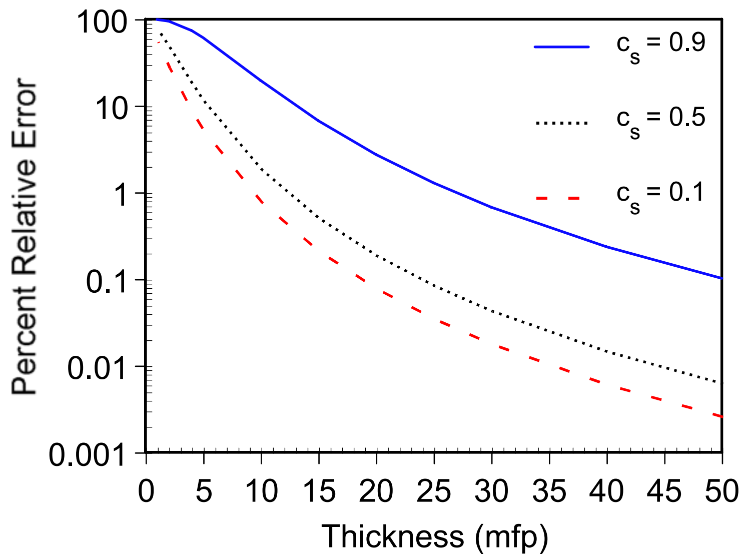

Figure 3. GFM percent relative error of dominance ratio approximation.

2.2. Spherical Geometry

Spherical geometry is somewhat simpler than Cartesian geometry for this analysis as the eigenfunctions are of a single function. In Cartesian geometry, the eigenfunctions are sines and cosines, where in spherical geometry, all eigenfunction are of the form sin(Br)/r – the eigenfunctions of the form cos(Br)/r are always eliminated to maintain a bounded value at the origin. Using one-group diffusion theory, we find that the eigenvalues for a homogeneous sphere of radius R are given by

$$k_n = \frac{\nu\Sigma_f / \Sigma_a}{1 + \frac{D}{\Sigma_a}\left(\frac{n\pi}{R+\delta}\right)^2} \quad , \quad n = 1, 2, 3, \dots \ . \tag{5}$$

The dominance ratio is therefore

$$DR = \frac{1 + \frac{D}{\Sigma_a}\left(\frac{\pi}{R+\delta}\right)^2}{1 + \frac{4D}{\Sigma_a}\left(\frac{\pi}{R+\delta}\right)^2} \quad , \tag{6}$$

which we can expand in a Taylor series as we did in Eq. (3) above. As with the homogeneous slab, if we take the *cube* of the ratio of the fundamental mode to the infinite multiplication factor and expand into a Taylor series, we find that it is equal to the expansion for the dominance ratio to two terms. Thus, we see that for a homogeneous sphere, as with the homogeneous slab, we can approximate the dominance ratio by taking the cube of the ratio of the effective multiplication to the infinite multiplication. This is not surprising given the equivalence of one-group homogeneous isotropically scattering slabs and spheres [4].

2.3. Cylindrical Geometry

The "fun" is lost for infinite cylindrical geometry because the eigenfunctions are Bessel functions and the eigenvalues are in terms of the zeros of Bessel functions, which are not integer multiples of each other (except in the limit of large order). As an example, the Taylor series expansions for the dominance ratio and k_{eff}/k_∞ are

$$DR = 1 + \frac{D}{\Sigma_a}\frac{x_1^2 - x_2^2}{(R+\delta)^2} + \left(\frac{D}{\Sigma_a}\right)^2 \frac{x_2^4 - x_1^2 x_2^2}{(R+\delta)^4} + \dots \quad , \tag{7}$$

$$\frac{k_{eff}}{k_\infty} = 1 - \frac{D}{\Sigma_a}\frac{x_1^2}{(R+\delta)^2} + \left(\frac{D}{\Sigma_a}\right)^2 \frac{x_1^4}{(R+\delta)^4} - \dots \quad , \tag{8}$$

where x_i are the zeros of the Bessel function $J_0(x)$ and R is the radius of the cylinder. Clearly, no clean correspondence is possible as all terms in the ratio k_{eff}/k_∞ are in terms of x_1 and all terms in the dominance ratio contain both x_1 and x_2. However, we can easily estimate the equivalent exponent for an infinite system. The first two terms of an expansion of $(k_{eff}/k_\infty)^n$ system will be of the form

$$\left(\frac{k_{eff}}{k_\infty}\right)^n \approx 1 - n\frac{D}{\Sigma_a}\frac{x_1^2}{(R+\delta)^2} \quad , \tag{9}$$

which when compared to the first two terms of Eq. (7), we see that $n \sim (x_2^2 - x_1^2)/x_1^2$, or 4.269 to obtain correspondence to the dominance ratio.

3. TWO-DIMENSIONAL ANALYSIS

The analysis becomes more complicated as we examine two-dimensional cases. Again, we assume homogeneous multiplying systems that scatter neutrons isotropically. We will consider Cartesian (square and rectangular) and cylindrical systems.

3.1. Cartesian Geometry - Square

Two-dimensional analysis (a homogeneous square of infinite dimension in the longitudinal direction) of the dominance ratio is complicated by the fact that there are three "sets" of eigenfunctions: cosine-cosine, sine-sine, and sine-cosine. The eigenvalues for a homogeneous square of side dimension a (in all combinations) are shown in Eqs. (10) with the extrapolation distances incorporated into the side dimension for convenience.

cos-cos $$k_{m,n} = \frac{\nu\Sigma_f/\Sigma_a}{1+\frac{D\pi^2}{4\Sigma_a a^2}\left[m^2+n^2\right]} \quad , \quad m, n = 1, 3, 5,\ldots \tag{10a}$$

sin-sin $$k_{m,n} = \frac{\nu\Sigma_f/\Sigma_a}{1+\frac{D\pi^2}{4\Sigma_a a^2}\left[4m^2+4n^2\right]} \quad , \quad m, n = 1, 2, 3, \ldots \tag{10b}$$

sin-cos $$k_{m,n} = \frac{\nu\Sigma_f/\Sigma_a}{1+\frac{D\pi^2}{4\Sigma_a a^2}\left[4m^2+n^2\right]} \quad , \quad m = 1, 2, 3, \ldots, n = 1, 3, 5,\ldots \tag{10c}$$

Note that there is a fourth set of eigenvalues in the form of cosine-sine, but the value of these eigenvalues will be the same as those from Eq. (10c). The dominance ratio, which happens to be the ratio of the first sine-cosine eigenvalue to the fundamental eigenvalue, is expanded in a Taylor series to give

$$DR = 1 - \frac{3}{4}\frac{D\pi^2}{\Sigma_a}\frac{1}{a^2} + \frac{15}{16}\left(\frac{D\pi^2}{\Sigma_a}\right)^2\frac{1}{a^4} - \ldots \quad . \tag{11}$$

If we take the ratio of the fundamental mode to the infinite multiplication factor to the 3/2 power and expand into a Taylor series, we find that

$$\left(\frac{k_{eff}}{k_\infty}\right)^{3/2} = 1 - \frac{3}{4}\frac{D\pi^2}{\Sigma_a}\frac{1}{a^2} + \frac{15}{32}\left(\frac{D\pi^2}{\Sigma_a}\right)^2\frac{1}{a^4} - \ldots \quad . \tag{12}$$

When we apply a reflecting boundary condition on a central plane of the square, i.e., a half-core, we obtain one set of eigenfunctions that is only a cosine in the direction perpendicular to the reflecting plane and a set of sines and cosines in the other direction. Thus, our eigenvalues are limited to those expressed in Eq. (10a) and Eq. (10c). Performing Taylor expansions of the related dominance ratio and various powers of k_{eff}/k_∞, we see that the dominance ratio is approximated by $(k_{eff}/k_\infty)^{3/2}$ for large homogeneous square systems with a single reflecting boundary condition.

If we use two reflecting planes in the square, i.e., we evaluate the quarter square with reflecting boundary conditions on two adjacent edges, we only retain the cosine eigenvalues per Eq. (10a). Performing Taylor expansions of the related dominance ratio and various powers of k_{eff}/k_∞, we see that the dominance ratio is now approximated by $(k_{eff}/k_\infty)^4$ for large homogeneous square systems with two reflecting boundary conditions.

As mentioned previously, large systems tend to have dominance ratios near one, which makes it difficult for Monte Carlo or discrete ordinates codes to obtain accurate results in a timely fashion. In the analysis above, the important result is that convergence should be faster for a fully reflected square system, i.e., a quarter-core, relative to an unreflected system as the dominance ratio of the fully reflected square is approximated by the factor $(k_{eff}/k_\infty)^4$, where the dominance ratio for the unreflected square is approximated by $(k_{eff}/k_\infty)^{3/2}$. The farther the dominance ratio is from one, the faster convergence should be obtained in a numerical calculation. Just as important is the fact that using a half-square (one reflecting plane) does not yield any enhanced convergence [both dominance ratios go as $(k_{eff}/k_\infty)^{3/2}$].

3.2. Cartesian Geometry - Rectangle

Analysis of rectangular geometry is similar to that of the square except that the number of possible unique eigenvalue forms is increased by one. We assume a rectangular system of dimension $a \times b$, where a is greater than b. Because the two side dimensions are different, we now have four eigenfunction combinations: cosine-cosine, sine-sine, cosine-sine, and sine-cosine. The eigenvalues for a homogeneous rectangle of side dimension $a \times b$ (in all combinations) are shown in Eqs. (13) (again, we do not explicitly show extrapolation distances for convenience).

cos-cos
$$k_{m,n} = \frac{\nu\Sigma_f / \Sigma_a}{1 + \frac{D\pi^2}{\Sigma_a}\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]} \quad , \quad m, n = 1, 3, 5, \ldots \tag{13a}$$

sin-sin
$$k_{m,n} = \frac{\nu\Sigma_f / \Sigma_a}{1 + \frac{D\pi^2}{\Sigma_a}\left[\frac{4m^2}{a^2} + \frac{4n^2}{b^2}\right]} \quad , \quad m, n = 1, 2, 3, \ldots \tag{13b}$$

sin-cos
$$k_{m,n} = \frac{\nu\Sigma_f / \Sigma_a}{1 + \frac{D\pi^2}{\Sigma_a}\left[\frac{4m^2}{a^2} + \frac{n^2}{b^2}\right]} \quad , \quad m = 1, 2, 3, \ldots, n = 1, 3, 5, \ldots \tag{13c}$$

cos-sin
$$k_{m,n} = \frac{\nu\Sigma_f / \Sigma_a}{1 + \frac{D\pi^2}{\Sigma_a}\left[\frac{m^2}{a^2} + \frac{4n^2}{b^2}\right]} \quad , \quad m = 1, 3, 5, \ldots, n = 1, 2, 3, \ldots \tag{13d}$$

The dominance ratio happens to be the ratio of the first sine-cosine eigenvalue to the fundamental eigenvalue, and the exponent that yields the standard approximation to the dominance ratio for large systems is given by

$$n = \frac{3b^2}{a^2 + b^2} \quad . \tag{14}$$

Note that if $a = b$, we reduce to the result from the square system. If we reflect the rectangle in the direction associated with the small side, we obtain the previously seen result where the exponent n is again given by Eq. (14), i.e., there is no decrease in the dominance ratio relative to the unreflected system. However, if we reflect in the direction associated with the longer side a, we see that

$$n = \begin{cases} \dfrac{8b^2}{a^2 + b^2} & , \quad a > \sqrt{\tfrac{8}{3}}b \\ \dfrac{3a^2}{a^2 + b^2} & , \quad \sqrt{\tfrac{8}{3}}b > a \end{cases} \quad . \tag{15}$$

Finally, if we reflect in both dimensions, we see that

$$n = \frac{8b^2}{a^2 + b^2} \quad . \tag{16}$$

In this case, only reflecting the long dimension yields a decrease in the dominance ratio. With the square, we did not obtain any benefit by reflecting in only one direction. Also, for all rectangular results, we reduce to the results for a square system when $a = b$.

3.3. Cylindrical Geometry – Finite Cylinder

A finite cylindrical system is somewhat less complicated than a rectangular system because the portion of the eigenfunction that is a Bessel function has only one form [the functions $Y_0(B_r r)$ are always eliminated to retain a bounded flux as r approaches zero]. The two forms of the eigenvalue are (where the x_i are the zeros of the Bessel function)

$$k_{i,n} = \frac{\nu\Sigma_f / \Sigma_a}{1 + \dfrac{D\pi^2}{\Sigma_a}\left[\dfrac{x_i^2}{R^2} + \dfrac{n^2\pi^2}{H^2}\right]} \quad , \quad n = 1, 3, 5, \ldots \quad , \tag{17a}$$

$$k_{i,n} = \frac{\nu\Sigma_f / \Sigma_a}{1 + \dfrac{D\pi^2}{\Sigma_a}\left[\dfrac{x_i^2}{R^2} + \dfrac{4n^2\pi^2}{H^2}\right]} \quad , \quad n = 1, 2, 3, \ldots \quad . \tag{17b}$$

The first eigenvalue above the fundamental is determined by the height-to-diameter ratio. Examining the two possible eigenvalues, we find that the first eigenfunction above the fundamental is a sine function if

$$\frac{x_2^2 - x_1^2}{3\pi^2} > \frac{R^2}{H^2} \quad \text{or} \quad 0.913H > R \quad ; \tag{18}$$

otherwise, it is the cosine. Eq. (18) indicates that most cylinders will have the sine function as part of the first higher mode – only short/fat cylinders (e.g., a spill) will have a cosine eigenfunction in the next mode. The dominance ratio for the most common cylinders is approximated by the effective multiplication and infinite multiplication according to $DR \sim (k_{eff}/k_\infty)^n$, with

$$n = \frac{3\pi^2 R^2}{x_1^2 H^2 + \pi^2 R^2} \quad . \tag{19}$$

If we reflect the cylinder in the axial direction, we obtain only cosines as eigenfunctions in the axial component, and the exponent is

$$n = \frac{8\pi^2 R^2}{x_1^2 H^2 + \pi^2 R^2} \quad . \tag{20}$$

Thus, we see a decrease in the dominance ratio by reflecting the system. For a right-circular cylinder with the height equal to the diameter, we see that the unreflected exponent is 0.299 and the reflected exponent is 0.797.

4. THREE-DIMENSIONAL ANALYSIS

Three-dimensional analysis of a cube complicates matters further with four "sets" of eigenfunctions: cosine-cosine-cosine, sine-sine-sine, sine-sine-cosine, and cosine-cosine-sine (as with the square, there are other eigenfunction forms that yield duplicate eigenvalues, e.g., an eigenfunction of the form sine-cosine-sine will yield the same eigenvalue as the associated sine-sine-cosine eigenfunction). The eigenvalues of a homogeneous cube of side length a (in combinations that yield distinct results) are shown in Eqs. (21).

cos-cos-cos
$$k_{\ell,m,n} = \frac{\nu\Sigma_f / \Sigma_a}{1 + \dfrac{D\pi^2}{4\Sigma_a a^2}\left[\ell^2 + m^2 + n^2\right]} \quad , \quad \ell, m, n = 1, 3, 5, \ldots \tag{21a}$$

sin-sin-sin
$$k_{\ell,m,n} = \frac{\nu\Sigma_f / \Sigma_a}{1 + \dfrac{D\pi^2}{4\Sigma_a a^2}\left[4\ell^2 + 4m^2 + 4n^2\right]} \quad , \quad \ell, m, n = 1, 2, 3, \ldots \tag{21b}$$

$$\text{sin-sin-cos} \quad k_{\ell,m,n} = \frac{\nu\Sigma_f / \Sigma_a}{1 + \dfrac{D\pi^2}{4\Sigma_a a^2}\left[4\ell^2 + 4m^2 + n^2\right]} \quad , \quad \ell, m = 1, 2, 3, \ldots , n = 1, 3, 5,\ldots \tag{21c}$$

$$\text{cos-cos-sin} \quad k_{\ell,m,n} = \frac{\nu\Sigma_f / \Sigma_a}{1 + \dfrac{D\pi^2}{4\Sigma_a a^2}\left[\ell^2 + m^2 + 4n^2\right]} \quad , \quad \ell, m = 1, 3, 5,\ldots , n = 1, 2, 3, \ldots \tag{21d}$$

Using analyses similar to the previous sections, we see that for an unreflected cube, the dominance ratio is given by the first cosine-cosine-sine eigenvalue divided by the fundamental. Expanding the dominance ratio in a Taylor series shows that it can be approximated by k_{eff}/k_∞ (i.e., with an exponent of $n = 1$). Placing a single reflecting plane in the cube (arbitrarily, we will assign the reflecting plane with the direction associated with the integer n above) yields the same result as the unreflected system, as does the case with two reflecting planes. In all three cases thus far, the first eigenvalue after the fundamental contains the integers associated with a cosine-cosine-sine eigenfunction. When the cube is fully reflected (one-eighth cube), we eliminate all sine eigenfunctions and obtain a smaller dominance ratio. In this case, we find that we can approximate the dominance ratio by $(k_{eff}/k_\infty)^{8/3}$.

As a summary, we show the asymptotic relationship between the dominance ratio and a power of the ratio k_{eff}/k_∞ in the form of Eq. (22) in Table II. These relationships hold for large homogeneous isotropically scattering systems.

$$DR \approx \left(\frac{k_{eff}}{k_\infty}\right)^n \quad , \tag{22}$$

An interesting phenomenon is evident from the results summarized in Table II. In all Cartesian cases and in the finite cylinder, the ratio of the exponent for full reflection relative to the unreflected system is 8/3. This is the result of the integers involved with higher mode eigenvalues whose eigenfunctions are sines and cosines.

As a final note, we have verified many of the eigenvalues and eigenfunctions with fission matrix results and production transport codes [5], [6]. Fission matrix analysis yields all the eigenvalues and eigenvectors of the fission matrix. Because we have focused on homogeneous systems, we obtain a direct correlation between fission matrix results and the diffusion results presented here.

Table II. Summary Approximation Results.

Case	*n*
1-D Slab	3
1-D Half-Slab	8
1-D Sphere	3
1-D Infinite Cylinder	4.269
2-D Square	3/2
2-D Square Half-Core	3/2
2-D Square Quarter-Core	4
2-D 2×1 Rectangle	3/5
2-D 2×1 Rectangle Short Side Reflected	3/5
2-D 2×1 Rectangle Long Side Reflected	8/5
2-D 2×1 Rectangle Fully Reflected	8/5
2-D Cylinder, $H = 2R$	0.299
2-D Cylinder Reflected, $H = 2R$	0.797
3-D Cube	1
3-D Cube Half-Core	1
3-D Cube Quarter-Core	1
3-D Cube Eighth-Core	8/3

5. TWO-GROUP AND ANISOTROPIC RESULTS

To obtain some experience with approximations to the dominance ratio in more complicated systems, we extend the analysis to two-group one-dimensional Cartesian geometry. We take advantage of the one-dimensional sphere-slab equivalence [4] to obtain the fundamental (slab) and first mode (sphere) eigenvalues using ONEDANT [5]. In addition, we use the PERICLES code [6] as it provides an estimate of the dominance ratio. We use the cross section set shown in Table III, which amounts to systems with $k_\infty = 1$. Fundamental eigenvalues are for slabs of 100 cm thickness and spheres of 100 cm diameter (for equivalence).

As we analyze the results shown in Table IV, we see that the approximations to the dominance ratio derived in the previous sections and summarized in Table II are quite accurate relative to calculated dominance ratios. Thus, we have evidence that the approximations derived here extend to multi-group calculations.

Table III. One-dimensional slab two-group cross sections.

Cross Section [cm^{-1}]	Group 1	Group 2
Σ_{ti}	0.5	1
$\Sigma_{si\to i}$	0.1	0.8
$\Sigma_{si\to j}$	0.35	0
$\nu\Sigma_{fi}$	0.05	0.2
Σ_{ai}	0.05	0.2
χ_i	1.0	0

Table IV. One-dimensional slab results.

Case	k_{eff}	DR	$(k_{eff}/k_\infty)^n$
1-D Slab - DANT	0.9970	0.9913	0.9912
1-D Sphere - DANT	0.9883	--	--
1-D Slab - PERICLES	0.9970	0.9912	0.9912
1-D Half-Slab - PERICLES	0.9970	0.9769	0.9766

Any significant degree of anisotropic scattering will negate an approximation using diffusion theory. However, for large systems, anisotropic scattering effects are reduced because leakage is reduced. As system size increases, the effects of anisotropic scattering are continuously reduced, to the point that the expression for the infinite multiplication of a homogeneous system is not a function of the scattering cross section. Thus, we expect that for large systems, the above approximations will hold provided the degree of anisotropic scattering is not uncommonly severe.

6. CONCLUSIONS

We have provided some interesting approximations to the dominance ratio for large systems using diffusion and transport theory formalisms. This work is intended to provide the nuclear community with a simple estimate of the dominance ratio for large systems. The drawback of this analysis is that we have limited our consideration to the most simple systems; however, we have provided some results to show that the approximations are valid for multi-group calculations and discuss their application relative to materials with anisotropic scattering cross sections. These results provide important information regarding the appropriate use of reflective boundary conditions, specifically where use of reflective boundary conditions will not improve the convergence rate of a calculation.

The primary extension of this analysis is obvious – we want to examine the applicability or any related approximations for heterogeneous geometries. It does not appear that simple approximations like those presented here will suffice for heterogeneous geometries. However, given that we are focused on large systems, we may be able to use some historical approximations that homogenize a heterogeneous system. As we see in this analysis, simple calculations can provide a good approximation to the dominance ratio – the same may be true for large heterogeneous systems. In addition to examining heterogeneous systems, we want to extend the above analysis from multi-group systems to continuous energy systems and show that these approximations hold in this most general case.

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