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ANALYTICAL SOLUTION OF THE NEUTRON DIFFUSION EQUATION IN THREE-DIMENSIONAL CYLINDRICAL GEOMETRY FOR APPLICATION IN A NODAL METHOD

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ABSTRACT

Rigorous analytical nodal methods have not previously been developed for general three-dimensional cylindrical geometry because the usual required transverse integration procedure does not work. The mathematics is presented here for a rigorous analytical nodal method that circumvents the transverse integration procedure. As in traditional nodal methods in multidimensional geometries, the solution of the multidimensional neutron diffusion equation is obtained in the form of a set of one-dimensional analytical transverse-averaged fluxes in each dimension. However, in the radial and azimuthal directions, these one-dimensional solutions are not obtained by transverse integration of the original multidimensional equation. Instead, a two-dimensional solution in r and θ is obtained for the z-averaged differential equation, and this two-dimensional solution is averaged separately over r and θ to obtain the required one-dimensional solutions. The axial solution is obtained by traditional transverse integration. The solutions are used to formulate a nodal method case in the coarse-mesh finite-difference framework.

Key words: nodal methods, cylindrical geometry.

1.0 INTRODUCTION

The usual strategy for solving the neutron diffusion equation in two or three dimensions by nodal methods is to reduce the multidimensional partial differential equation to a set of ordinary differential equations (ODEs) in the separate spatial coordinates. This reduction is accomplished by "transverse integration" of the equation [1,2]. For example, in three-dimensional Cartesian coordinates, the three-dimensional equation is first integrated over x and y to obtain an ODE in z, then over x and z to obtain an ODE in y, and finally over y and z to obtain an ODE in x. Then the ODEs are solved to obtain one-dimensional solutions for the neutron fluxes averaged over the other two dimensions ("transverse-averaged fluxes"). These solutions are found in regions ("nodes") in which average material properties and cross sections can be adequately defined and obtained [3]. Because the solution in each node is an exact analytical solution, the nodes can be much larger than the mesh elements used in finite-difference solutions. Then the solutions in the different nodes are coupled by applying interface conditions, ultimately fixing the solutions to the external boundary conditions.

However, the transverse integration procedure fails in (r, θ) or (r, θ, z) cylindrical geometry, because the transverse integration over r (in 2-d) or z and r (in 3-d) leads to an impasse, as shown in Section 4.0. In this paper, it is shown how the impasse can be circumvented. The diffusion equation is readily integrated over z to obtain an equation in r and θ for the z-averaged neutron flux ${}^{z}\overline{\phi}(r,\theta)$. Then the solution for ${}^{z}\overline{\phi}(r,\theta)$ is found analytically and integrated over each remaining coordinate to obtain a solution for the neutron flux averaged over the other coordinate (and z). Even though the solution for ${}^{z}\overline{\phi}(r,\theta)$ has been found, it is still necessary to compute the one-dimensional solutions, because it is less practical to couple the two-dimensional solutions across node interfaces.

Thus, instead of obtaining one-dimensional differential equations and solving each of them to obtain one-dimensional solutions, one finds a two-dimensional solution directly and then integrates it to obtain one-dimensional solutions. The two-dimensional solution for ${}^{z}\overline{\phi}(r,\theta)$ has been found by three different methods: the method of integral transforms, the method of separation of variables, and the Green's function method. In this paper, the details of the solutions are not provided. Here, only a sketch of the Green's function solution is given. The approach is similar to that previously taken by Burns in Cartesian coordinates [4].

To complete the three-dimensional solution, one must obtain and solve an ODE in z. The usual transverse integration procedure works in this case, because the integration over θ is carried out first.

2.0 ACCOUNTING FOR TIME-DEPENDENCE

The time-dependent multigroup neutron diffusion equation in cylindrical geometry, in a typical node, is

$$\frac{1}{v_g}\frac{\partial \phi_g}{\partial t} - D_g\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi_g}{\partial \theta^2} + \frac{\partial^2 \phi_g}{\partial z^2}\right] + \Sigma_{Rg}\phi_g = S_g \quad , \tag{1}$$

where Σ_{Rg} = group removal cross section
S_g = group volumetric source rate (including inscattering)
D_g = group diffusion coefficient,
and the other symbols have their usual meaning in nuclear reactor physics.

The Laplace transform of Eq. (1) is

$$D_g\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{\phi}_g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \tilde{\phi}_g}{\partial \theta^2} + \frac{\partial^2 \tilde{\phi}_g}{\partial z^2}\right] - \Sigma_{Rg}\tilde{\phi}_g - \frac{1}{v_g}s\tilde{\phi}_g = -\frac{S_g}{s} - \frac{\phi_g(0)}{v_g} \quad , \tag{2}$$

where $\tilde{\phi}_g = \mathcal{L}(\phi_g) = \int_{t=0}^{\infty} e^{-st}\phi_g(t)dt$ is the Laplace transform of ϕ_g, and where $\phi_g(0)$ is the flux $\phi_g(r,\theta,z,t)$ evaluated at t=0.

Next, define

$$\tilde{\Sigma}_{Rg} = \Sigma_{Rg} + \frac{s}{v_g} \tag{3a}$$

and

$$\tilde{S}_g = \frac{S_g}{s} + \frac{\phi_g(0)}{v_g}. \tag{3b}$$

Then Eq. (2) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{\phi}_g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\tilde{\phi}_g}{\partial\theta^2} + \frac{\partial^2\tilde{\phi}_g}{\partial z^2} - \frac{\tilde{\Sigma}_{Rg}\tilde{\phi}_g}{D_g} = -\frac{\tilde{S}_g}{D_g}. \tag{4}$$

This equation has exactly the same form as the time-independent equation obtained from Eq. (1), but with modified definitions for the removal cross section and the source term. Therefore, the same procedure can be applied both to time-dependent and steady problems.

3.0 TRANSVERSE INTEGRATION OVER *z*

The transverse integration procedure begins with the multigroup diffusion equation given by Eq. (4) or its steady-state untransformed equivalent (written here with the energy group index omitted):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\Sigma_R\phi}{D} = -\frac{S}{D}, \tag{5}$$

This equation is first integrated over $z_k \le z \le z_{k+1}$, the domain of z in the node, to obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\, {}^z\overline{\phi}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\, {}^z\overline{\phi}}{\partial\theta^2} - \frac{\Sigma_R\, {}^z\overline{\phi}}{D} = \frac{L_z}{D} - \frac{{}^z\overline{S}}{D}, \tag{6}$$

where

$${}^z\overline{\phi}(r,\theta) \equiv \frac{1}{z_{k+1}-z_k}\int_{z_k}^{z_{k+1}} \phi(r,\theta,z)dz$$

and

$${}^z\overline{S} \equiv \frac{1}{z_{k+1}-z_k}\int_{z_k}^{z_{k+1}} S\,dz$$

are the z-averaged neutron flux and neutron source, and

$$L_z(r,\theta) \equiv \frac{1}{z_{k+1} - z_k}\left[J_z(r,\theta,z_{k+1}) - J_z(r,\theta,z_k)\right]$$

is the z-directed transverse leakage for the diffusion equation in r and θ.

4.0 THE FAILURE OF TRADITIONAL TRANSVERSE INTEGRATION IN *r*

The final step in producing a θ-dependent equation, were it possible, would be to integrate Eq. (6) over $r_i \le r \le r_{i+1}$, the domain of r in the node. The appropriate average over r includes the weighting factor r to account for the geometry.

$$\int_{r_i}^{r_{i+1}} \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial\, {}^z\overline{\phi}}{\partial r} \right) + \frac{1}{r^2}\left(\frac{\partial^2\, {}^z\overline{\phi}}{\partial \theta^2} \right) - \frac{\Sigma_R}{D}\, {}^z\overline{\phi} \right] r\, dr = \int_{r_i}^{r_{i+1}} \left(\frac{L_z}{D} - \frac{{}^z\overline{S}}{D} \right) r\, dr \, . \tag{7}$$

A difficulty arises with the integration of the azimuthal term,

$$\mathrm{I}^{(1)}_{Az} \equiv \int_{r_i}^{r_{i+1}} \frac{1}{r}\left(\frac{\partial^2\, {}^z\overline{\phi}}{\partial \theta^2} \right) dr \, . \tag{8}$$

The goal of the transverse integration process is to obtain an equation in the θ-dependent r- and z-averaged flux, defined by

$$ {}^{rz}\overline{\phi}(\theta) \equiv \frac{2}{r_{i+1}^2 - r_i^2} \int_{r_i}^{r_{i+1}} {}^z\overline{\phi}\, r\, dr \, . \tag{9}$$

But the presence of $1/r$ in the integrand of Eq. (8) makes this goal unattainable. Successive integration by parts does not work, because a logarithm is obtained that prevents the eventual elimination of r-dependent factors in the integrand.

The use of other weighting factors (higher powers of r) was shown to result in similar failures.

5.0 SOLUTION BY THE GREEN'S FUNCTION METHOD

Of the three solutions obtained, as mentioned in Section 1.0, the Green's function solution yields the most efficient implementation. Therefore, only the Green's function solution is discussed in this paper.

It is convenient to subsume the transverse-leakage term in Eq. (6) into an effective source term:

$$\frac{L_z}{D} - \frac{{}^z\overline{S}}{D} \equiv S(r,\theta)\,. \tag{10}$$

Then Eq. (6) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\,{}^z\overline{\phi}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\,{}^z\overline{\phi}}{\partial\theta^2} - \frac{\Sigma_R\,{}^z\overline{\phi}}{D} = S(r,\theta)\,, \tag{11}$$

subject to the boundary conditions

$$-D\frac{\partial\,{}^z\overline{\phi}}{\partial r}\bigg|_{r=r_i} = {}^z\overline{J}_r(r_i,\theta) \tag{12a}$$

$$-D\frac{\partial\,{}^z\overline{\phi}}{\partial r}\bigg|_{r=r_{i+1}} = {}^z\overline{J}_r(r_{i+1},\theta) \tag{12b}$$

$$-\frac{D}{r}\frac{\partial\,{}^z\overline{\phi}}{\partial\theta}\bigg|_{\theta=\theta_j} = {}^z\overline{J}_\theta(r,\theta_j) \tag{12c}$$

and

$$-\frac{D}{r}\frac{\partial\,{}^z\overline{\phi}}{\partial\theta}\bigg|_{\theta=\theta_{j+1}} = {}^z\overline{J}_\theta(r,\theta_{j+1})\quad, \tag{12d}$$

in which the boundary currents are averaged over z in analogy with the definitions for ${}^z\overline{\phi}$ and ${}^z\overline{S}$ following Eq. (6).

The solution of Eq. (11) is to be generated from the Green's function, $G(r,\theta;\, r_o,\theta_o)$, which satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 G}{\partial\theta^2} - k^2 G = \frac{\delta(r-r_o)}{r}\delta(\theta-\theta_o) \tag{13}$$

subject to the homogeneous analogues of Eqs. (12); in Eq. (13), $k = \sqrt{\Sigma_R / D}$.

Standard methods for applying the Green's function technique are used to obtain

$$G^I(r,\theta;r_\circ,\theta_\circ) = \sum_{m=0}^{\infty}\frac{2-\delta_{m0}}{\theta_{j+1}-\theta_j}\cos\left[\frac{m\pi(\theta-\theta_j)}{\theta_{j+1}-\theta_j}\right]\cos\left[\frac{m\pi(\theta_\circ-\theta_j)}{\theta_{j+1}-\theta_j}\right]\frac{\left[I_\nu(kr)K_\nu'(kr_i)-I_\nu'(kr_i)K_\nu(kr)\right]}{H_\nu(k,r_i,r_{i+1})}$$

$$\times\left[I_\nu(kr_o)K_\nu'(kr_{i+1})-I_\nu'(kr_{i+1})K_\nu(kr_o)\right], \tag{14a}$$

which is valid in the domain $r_i \le r \le r_o$, and

$$G^{II}(r,\theta;r_o,\theta_o)=\sum_{m=0}^{\infty}\frac{2-\delta_{m0}}{\theta_{j+1}-\theta_j}\cos\left[\frac{m\pi(\theta-\theta_j)}{\theta_{j+1}-\theta_j}\right]\cos\left[\frac{m\pi(\theta_\circ-\theta_j)}{\theta_{j+1}-\theta_j}\right]\frac{\left[I_\nu(kr)K_\nu'(kr_{i+1})-I_\nu'(kr_{i+1})K_\nu(kr)\right]}{H_\nu(k,r_i,r_{i+1})}$$
$$\times\left[I_\nu(kr_o)K_\nu'(kr_i)-I_\nu'(kr_i)K_\nu(kr_o)\right], \tag{14b}$$

which is valid in the domain $r_o \le r \le r_{i+1}$, where $H_\nu(k,r_i,r_{i+1})$ is a known function that depends only on the composition and the geometry, δ_{mo} is the Kronecker delta function, $I_\nu(kr)$ and $K_\nu(kr)$ are hyperbolic Bessel functions of order ν , and $\nu = m\ell/2$, in which ℓ is the number of (equal) angular zones into which the reactor is nodalized. The prime (') denotes a derivative with respect to the entire argument. Equations (14) are singular at the origin, so instead of a radially and azimuthally dependent solution at the origin, a radially and azimuthally uniform solution is found in a small node around the origin.

Further standard procedures, together with the use of the homogeneous boundary conditions on G and the inhomogeneous boundary conditions on ${}^z\overline{\phi}$ (Eqs. (12)), yield the final result

$${}^z\overline{\phi}(r,\theta)=\int_{r_o=r_i}^{r_{i+1}}\int_{\theta_o=\theta_j}^{\theta_{j+1}}G(r,\theta;r_o\theta_o)S(r_o,\theta_o)r_o dr_o d\theta_o+\int_{r_o=r_i}^{r_{i+1}}\frac{G(r,\theta;r_o,\theta_{j+1})}{D}\,{}^z\overline{J}_\theta(r_o,\theta_{j+1})\,dr_o$$
$$+\int_{\theta_o=\theta_j}^{\theta_{j+1}}\frac{G(r,\theta;r_{i+1},\theta_o)}{D}\,{}^z\overline{J}_r(r_{i+1},\theta_o)r_{i+1}d\theta_o-\int_{r_o=r_i}^{r_{i+1}}\frac{G(r,\theta;r_o,\theta_j)}{D}\,{}^z\overline{J}_\theta(r_o,\theta_j)\,dr_o$$
$$-\int_{\theta_o=\theta_j}^{\theta_{j+1}}\frac{G(r,\theta;r_i,\theta_o)}{D}\,{}^z\overline{J}_r(r_i,\theta_o)\,r_i\,d\theta_o\,. \tag{15}$$

6.0 DERIVATION OF TRANSVERSE-AVERAGED FLUXES

The next step in the solution procedure is to integrate Eq. (15) over r and θ separately to obtain one-dimensional solutions for the transverse-averaged nodal fluxes. This integration process is straightforward but very tedious because of the presence of the hyperbolic Bessel functions in Eqs. (14). Some of the integrals can be carried out analytically, but even for these integrals it is more efficient computationally to evaluate them numerically by nested Gaussian quadrature.

The resulting expressions for the one-dimensional fluxes are very lengthy and cumbersome, so because of space limitations they are not presented here in full. Instead, the expressions are presented in a condensed form, with selected coefficients presented in detail. This presentation shows the kinds of integrals that need to be evaluated by the nested quadrature procedure. The solutions are obtained in turn for the r-dependent and θ-dependent transverse-integrated fluxes.

6.1 Derivation of *r*-Dependent One-Dimensional Transverse-Integrated Flux

The *r*-dependent flux, resulting from the integration of Eq. (15) over θ, takes the form

$$^{\theta z}\overline{\phi}(r) = g_1(r)\left[\,^{r\theta}\overline{J}_z(z_{k+1}) - \,^{r\theta}\overline{J}_z(z_k)\right] - g_2(r) + g_3(r)\left[\,^{rz}\overline{J}_\theta(\theta_{j+1}) - \,^{rz}\overline{J}_\theta(\theta_j)\right]$$

$$+g_4(r)\,^{\theta z}\overline{J}_r(r_{i+1}) - g_5(r)\,^{\theta z}\overline{J}_r(r_i)\,, \qquad (16)$$

in which it has been assumed that the net currents on the boundaries are "flat" – i.e., $J_z(r,\theta,z_k) = \,^{r\theta}\overline{J}_z(z_k)$, etc., and the neutron source has been expanded in a series of Legendre polynomials, truncated after three terms (more terms may easily be added if necessary).

On the right side of Eq. (16), the first term accounts for the axial transverse leakage, $g_2(r)$ is the actual source term (not the effective source defined in Eq. (10)), the third term accounts for the azimuthal transverse leakage, and the last two terms account for the boundary currents on the radial surfaces. Only one of the coefficients $g_n(r)$ is presented below; the others contain similar combinations of sums, products, and integrals of the hyperbolic Bessel functions. The coefficient chosen for display is the azimuthal transverse-leakage coefficient:

$$g_3(r) = \frac{1}{DH_o(\theta_{j+1} - \theta_j)}\left[\ell_1(r)I_o(kr) + \ell_2(r)K_o(kr)\right], \qquad (17a)$$

in which

$$\ell_1(r) = K_o'(kr_i)K_o'(kr_{i+1})\,i_{I0} - I_o'(kr_i)K_o'(kr_{i+1})\,i_{K10}(r) - K_o'(kr_i)I_o'(kr_{i+1})\,i_{K11}(r)\,, \qquad (17b)$$

with

$$i_{I0} = \int_{r_o=r_i}^{r_{i+1}} I_o(kr_o)dr_o\,, \qquad (17c)$$

$$i_{K10}(r) = \int_{r_o=r_i}^{r} K_o(kr_o)dr_o\,, \qquad (17d)$$

and

$$i_{K11}(r) = \int_{r_o=r}^{r_{i+1}} K_o(kr_o)dr_o\,, \qquad (17e)$$

and $\ell_2(r)$ is similar in form, with different integrals appearing. It has been confirmed that in the azimuthally symmetric problem, Eqs. (16) and (17) are equivalent to a previously published solution for *r-z* cylindrical geometry [5].

6.2 Derivation of the θ-Dependent One-Dimensional Transverse-Integrated Flux

In the derivation above for the θ-averaged flux, ${}^{\theta z}\overline{\phi}(r)$, all the trigonometric functions except the fundamental one integrate to zero, so all the infinite series in the integrals in Eq. (15), using Eqs. (14) for the Green's function, disappear. However, in the calculation of the r-averaged flux, ${}^{rz}\overline{\phi}(\theta)$, the infinite series remain in the source term and in the terms involving the boundary currents in the θ-direction. Because the functions I_ν become extremely small and the functions K_ν become extremely large as $\nu \rightarrow \infty$, computational underflows and overflows occur when the individual hyperbolic Bessel functions and their derivatives and integrals are evaluated for large values of ν. When the expressions for G in Eqs. (14) are expanded for integration as done above for the average over θ, evaluation of individual Bessel functions at large values of ν becomes necessary, and the numerical evaluation of the flux fails because of these numerical underflows and overflows. However, if the integrals are evaluated numerically with the Green's function G left in the form of Eqs. (14), after a few terms the hyperbolic Bessel functions may be expressed in asymptotic expansions, and it is found that the exponential factors in these expansions, which account for the large or small magnitude of the Bessel functions, cancel out and the remaining factors are tractable in magnitude for computation. On the other hand, in the terms of the r-averaged flux expression that retain only the fundamental trigonometric function, it is found that fewer computational operations are required in numerical integration if the multiplications in the expressions for G are carried out; thus, in the expression below for ${}^{rz}\overline{\phi}(\theta)$, each term appears in its most convenient form for numerical evaluation. The resulting expression for the r-averaged flux is

$$
\begin{aligned}
{}^{rz}\overline{\phi}(\theta) = {} & \frac{2}{(r_{i+1}^2 - r_i^2)(z_{k+1} - z_k)DH_\circ}\left[J_z(z_{k+1}) - J_z(z_k)\right]\Big[K'_\circ(kr_{i+1})K'_\circ(kr_i)\xi^{(\circ)}_{II4} - K'_\circ(kr_{i+1})I'_\circ(kr_i)\xi^{(\circ)}_{IK4} \\
& - I'_\circ(kr_{i+1})K'_\circ(kr_i)\xi^{(\circ)}_{KI4} + I'_\circ(kr_{i+1})I'_\circ(kr_i)\xi^{(\circ)}_{KK4} + K'_\circ(kr_i)K'_\circ(kr_{i+1})\xi^{(\circ)}_{II7} - K'_\circ(kr_i)I'_\circ(kr_{i+1})\xi^{(\circ)}_{IK7} \\
& - I'_\circ(kr_i)K'_\circ(kr_{i+1})\xi^{(\circ)}_{KI7} + I'_\circ(kr_i)I'_\circ(kr_{i+1})\xi^{(\circ)}_{KK7}\Big] - \frac{2}{D(r_{i+1}^2 - r_i^2)}\int_{r=r_i}^{r_{i+1}}\int_{r_\circ=r_i}^{r_{i+1}}\int_{\theta_\circ=\theta_j}^{\theta_{j+1}} G(r,\theta;r_\circ,\theta_\circ)\,{}^{z}\overline{S}(r_\circ,\theta_\circ)\,r r_\circ\, dr_\circ\, d\theta_\circ\, dr \\
& + \frac{2}{(r_{i+1}^2 - r_i^2)D} J_\theta(\theta_{j+1})\int_{r=r_i}^{r_{i+1}}\int_{r_\circ=r_i}^{r_{i+1}} G(r,\theta;r_\circ,\theta_{j+1})\,r\,dr_\circ\, dr - \frac{2}{(r_{i+1}^2 - r_i^2)D} J_\theta(\theta_j)\int_{r=r_i}^{r_{i+1}}\int_{r_\circ=r_i}^{r_{i+1}} G(r,\theta;r_\circ,\theta_j)\,r\,dr_\circ\, dr \\
& + \frac{2}{(r_{i+1}^2 - r_i^2)D}\frac{r_{i+1}}{H_\circ}\left[I_\circ(kr_{i+1})K'_\circ(kr_{i+1}) - I'_\circ(kr_{i+1})K_\circ(kr_{i+1})\right]\left[K'_\circ(kr_i)i_{I1} - I'_\circ(kr_i)i_{K1}\right]J_r(r_{i+1}) \\
& - \frac{2}{(r_{i+1}^2 - r_i^2)D}\frac{r_i}{H_\circ}\left[I_\circ(kr_i)K'_\circ(kr_i) - I'_\circ(kr_i)K_\circ(kr_i)\right]\left[K'_\circ(kr_{i+1})i_{I1} - I'_\circ(kr_{i+1})i_{K1}\right]J_r(r_i)
\end{aligned}
\qquad (18a)
$$

where, for example,

$$\xi_{IK4}^{(\circ)} = \int_{r_\circ = r_i}^{r_{i+1}} I_\circ(kr_\circ) i_{K4}^{(\circ)}(r_\circ) r_\circ dr_\circ \tag{18b}$$

in which

$$i_{K4}^{(\circ)}(r_\circ) = \int_{r=r_i}^{r_\circ} K_\circ(kr) r dr \,, \tag{18c}$$

and where

$$i_{I1} = \int_{r_\circ = r_i}^{r_{i+1}} I_\circ(kr_\circ) r_\circ dr_\circ \;. \tag{18d}$$

6.3 Derivation of the *z*-Dependent One-Dimensional Transverse-Integrated Flux

The z-dependent flux is obtained by familiar methods; the only feature unique to general cylindrical geometry is the presence of the arc $\theta_j \leq \theta \leq \theta_{j+1}$ in some terms in the solution. The z-dependent flux takes a form identical to that of one-dimensional solutions in Cartesian geometry except for the presence of azimuthal and radial transverse leakages, which have different form from the transverse leakages in Cartesian coordinates.

7.0 IMPLEMENTATION IN THE COARSE-MESH FINITE-DIFFERENCE FRAMEWORK

The solution described above can be implemented in a purely nodal formulation, but it is convenient to exploit the structure of an existing finite-difference computer code in the well-known coarse-mesh finite-difference (CMFD) method [6,7,8,9]. Since the three-dimensional finite-difference version of the INEEL's PEBBED code [10] for pebble-bed reactors (PBRs) has been developed, this new nodal solution is being implemented in the CMFD formulation within PEBBED.

The basis of the CMFD method chosen for this application is the seven-point difference equation of the three-dimensional finite-difference solution, modified with correction coefficients to force the fluxes obtained from the difference equation to contain the information from the nodal solution. Derivation of the difference equation follows Stamm'ler and Abbate [11]. Derivation of the corrected equation follows Sutton [8]. In cylindrical geometry, the corrected difference equation for the node-averaged fluxes is

$$\frac{-2r_i d^r_{i-1,i}}{r^2_{i+1}-r^2_i}\left(1-C^r_{ijk}\right)\bar{\phi}_{(i-1)jk} - \frac{2d^\theta_{j-1,j}}{\alpha_j (r_{i+1}+r_i)^2}\left(1-C^\theta_{ijk}\right)\bar{\phi}_{i(j-1)k} - \frac{d^z_{k-1,k}}{2a_z}\left(1-C^z_{ijk}\right)\bar{\phi}_{ij(k-1)}$$

$$+\left[\frac{2r_i d^r_{i-1,i}}{r^2_{i+1}-r^2_i}\left(1+C^r_{ijk}\right)+\frac{2d^\theta_{j-1,j}}{\alpha_j (r_{i+1}+r_i)^2}\left(1+C^\theta_{ijk}\right)+\frac{d^z_{k-1,k}}{2a_z}\left(1+C^z_{ijk}\right)+\frac{2r_{i+1} d^r_{i,i+1}}{r^2_{i+1}-r^2_i}\left(1-C^r_{(i+1)jk}\right)\right.$$

$$\left.+\frac{2d^\theta_{j,j+1}}{\alpha_j (r_{i+1}+r_i)^2}\left(1-C^\theta_{i(j+1)k}\right)+\frac{d^z_{k,k+1}}{2a_z}\left(1-C^z_{ij(k+1)}\right)+\Sigma_R\right]\bar{\phi}_{ijk} - \frac{2r_{i+1} d^r_{i,i+1}}{r^2_{i+1}-r^2_i}\left(1+C^r_{(i+1)jk}\right)\bar{\phi}_{(i+1)jk}$$

$$- \quad \frac{2d^\theta_{j,j+1}}{\alpha_j (r_{i+1}+r_i)^2}\left(1+C^\theta_{i(j+1)k}\right)\bar{\phi}_{i(j+1)k} - \frac{d^z_{k,k+1}}{2a_z}\left(1+C^z_{ij(k+1)}\right)\bar{\phi}_{ij(k+1)} = \bar{Q}_{ijk}\,, \tag{19}$$

where

$$2\alpha_j = \theta_{j+1} - \theta_j\,,$$

$$2a_z = z_{k+1} - z_k\,,$$

the quantities such as $d^r_{i-1,i}$ are hybrid diffusion coefficients of form

$$d^r_{i-1,i} = \frac{2d^r_{i-1}d^r_i}{d^r_{i-1}+d^r_i}\,,$$

in which $d^r_i = D_{ijk}/(r_{i+1}-r_i)$, etc., and the triple subscripts such as *ijk* denote the node in the region such as $r_i \le r \le r_{i+1}$, $\theta_j \le \theta \le \theta_{j+1}$, $z_k \le z \le z_{k+1}$.

As in previous work [8], the correction coefficients are given by equations of form

$$C^r_{ijk} = \frac{-{}^{\theta z}\bar{J}_r(r_i)/d^r_{i-1,i} + \bar{\phi}_{(i-1)jk} - \bar{\phi}_{ijk}}{\bar{\phi}_{ijk} + \bar{\phi}_{(i-1)jk}}\,, \tag{20}$$

where ${}^{\theta z}\bar{J}_r(r_i)$ is the *r*-component of the current averaged over the θ- and *z*-directions on the boundary at $r=r_i$.

The interface currents in the equations like Eq. (20) are found by performing a two-node solution at each interface, in which the latest values are used for the currents at the boundaries of the nodes other than the shared boundary. The interface current in the *r*-direction on the boundary of node *ijk* at $r=r_i$ is

$$\bar{J}_r^{ijk}(r_i) = \frac{1}{g_4^{(i-1)jk}(r_i) + g_5^{ijk}(r_i)} \Big\{ g_1^{ijk}(r_i)\Big[\bar{J}_z^{ijk}(z_{k+1}) - \bar{J}_z^{ijk}(z_k)\Big] - g_1^{(i-1)jk}(r_i)\Big[\bar{J}_z^{(i-1)jk}(z_{k+1}) - \bar{J}_z^{(i-1)jk}(z_k)\Big]$$

$$-g_2^{ijk}(r_i) + g_2^{(i-1)jk}(r_i) + g_3^{ijk}(r_i)\Big[\bar{J}_\theta^{ijk}(\theta_{j+1}) - \bar{J}_\theta^{ijk}(\theta_j)\Big] - g_3^{(i-1)jk}(r_i)\Big[\bar{J}_\theta^{(i-1)jk}(\theta_{j+1}) - \bar{J}_\theta^{(i-1)jk}(\theta_j)\Big]$$

$$+g_4^{ijk}(r_i)\bar{J}_r^{ijk}(r_{i+1}) + g_5^{(i-1)jk}(r_i)\bar{J}_r^{(i-1)jk}(r_{i-1})\Big\} \ , \qquad (21)$$

where the *g* coefficients are as defined in Eqs. (16), and the superscripts specify the node in which they are evaluated. In Eq. (21), the superscripts on the currents are omitted which specify the directions over which the averages are taken, because the subscript suffices to specify these directions unambiguously. That is, whichever variable the subscript specifies, the average is taken over the other two. Then the superscript states the node on whose boundaries the currents are evaluated, just as it specifies the node for the *g* coefficients. The expressions for the interface currents in the other two directions are similar to Eq. (22).

The solution of the system of seven-point difference equations, the equations for the correction coefficients, and the interface currents is accomplished by an iterative scheme. This solution is presently being implemented in the PEBBED code.

8.0 CONCLUSIONS

The transverse-integration approach has been traditionally applied for developing modern nodal methods for solution of the neutron diffusion equation of reactor physics. The method succeeds in the derivation of an ODE in *z*. It also succeeds in the derivation of an ODE in *r* (but this is not useful for the solution reported here). However, the method fails in the derivation of an ODE in θ.

In lieu of obtaining an ODE in θ by integrating over *r* the partial differential diffusion equation in *r* and θ (already averaged over *z*), the novel tactic is applied of finding a solution in *r* and θ directly and then integrating that solution over *r* to obtain the solution in θ. The two-dimensional solution is also integrated over θ to obtain a one-dimensional solution in *r*.

The Green's function solution and the coupled solution of the axial equation are being implemented in the CMFD formulation in the INEEL's PEBBED code for PBRs. It will also be implemented separately in the CYNOD code. This code and the numerical results obtained from it will be described in a future paper.

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