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KINETIC APPROXIMATIONS AND NOISE THEORY IN SOURCE-DRIVEN SUBCRITICAL SYSTEMS

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ABSTRACT

Some basic features of the dynamics of source-driven systems, as well as certain classical methods of calculating the space-dependent dynamics of such systems, are described in this paper. The applicability and adaptation of two methods for source driven systems, used also in the kinetics of critical systems, are analysed in some detail. These are the eigenfunction expansion and the flux factorisation method, the latter serving as the basis for the reactor kinetic approximations. The emphasis is on the physical meaning of the various terms of the solutions, and on the differences between the source-free and the source-driven cases. It is found that the use of these methods differ significantly between the two systems, corresponding to the different physics behind.

Key Words: Kinetic approximations, source-driven systems, neutron noise, Henry factorisation, point kinetics, adiabatic approximation

1. INTRODUCTION

There are several classical methods that have long been used for the description of the kinetics and the dynamics of critical or non-critical but source-free systems. Two such classical methods are the eigenfunction expansion and the so-called kinetic approximations (point kinetic, adiabatic, quasistatic), the latter being based on the flux factorisation (Henry factorisation) ([1], [2]) into an amplitude and a shape function. There exist of course numerous other, purely numerical schemes for the solution of the space-time-energy dependent neutronic equations, but the power of the classical methods is that they lend physical insight into the behaviour of the system. This is because certain terms of the solution dominate asymptotically as a function of some parameter, and moreover it is very easy to develop an intuitive feeling for such terms, such as the effect of a reactivity change, or the adiabatic distortion of the flux shape etc. This is especially true for the case of neutron noise, induced by stationary and small fluctuations of the cross sections, which will be one of the cases that we investigate.

The dynamic behaviour of source-driven subcritical systems has recently become interesting due to the increasing popularity of the concept of accelerator-driven subcritical systems (ADS). It is obvious that these systems have physical properties that are different from those of critical or source-free systems. Therefore, not surprisingly, the mathematical tools to be used for their description need to be modified. Moreover their properties, including their applicability, will also be different from those in source-free systems.

The purpose of this paper is to investigate two classical methods for the description of source-driven systems. One is the eigenfunction expansion, and the other is the kinetic approximations, based on the flux factorisation technique. The physical situations that we investigate by both of these methods will also consist of two simple basic cases. One is the attainment of the asymptotic behaviour of the flux in a steady system from an arbitrary initial flux shape; the other is the space and frequency dependent response of a system for small, stationary fluctuations of the cross sections. The emphasis is on the physical meaning of the solutions, and on the differences between the source-free and the source-driven cases. The source-driven subcritical case will be described by an extraneous source within the core. This is an approximation used to simplify the situation. Other descriptions are also possible, such as separating the source from the core and imposing the effect of the source through coupling at the interface [3]. The emphasis is however in both descriptions on the fact that the system is source- (or boundary condition)-driven, with relatively loose coupling between different spatial points of the system. This will determine much of the physical properties.

2. THE EIGENFUNCTION EXPANSION

Throughout the paper, we shall use one-group diffusion theory, and one group of delayed neutrons, to keep the reasoning simple. Extension to more complicated cases is straightforward. In what follows, we shall treat two simple cases of time- (or frequency)-dependent solutions: the approaching of the asymptotic state of the flux in a stationary system, and the neutron noise induced in the system by small fluctuations of the cross sections.

2.1 Asymptotic solutions in a steady system

Source-free system

In a source-free, critical or non-critical system, the diffusion equation can be written symbolically as

$$\frac{1}{v}\frac{\partial \phi(\boldsymbol{r}, t)}{\partial t} = \boldsymbol{L} \cdot \phi(\boldsymbol{r}, t) \tag{1}$$

The diffusion operator $\boldsymbol{L}$ is assumed to have a complete orthogonal set of eigenfunctions $\phi_n(\boldsymbol{r})$. Using the concept of the α-eigenfunctions [2], these are defined as

$$\boldsymbol{L}\phi_n(\boldsymbol{r}) = \alpha_n \phi_n(\boldsymbol{r}) \tag{2}$$

Actually in this notation it is implicitly assumed that the delayed neutrons are neglected. It is easy to extend the formalism to include even these, but for the simple argument we wan to make here it is not necessary. It is straightforward to show that with the use of the expansion (2), the solution of (1), for any given initial condition

$$\phi(\boldsymbol{r}, t=0) \equiv \phi(\boldsymbol{r}) \tag{3}$$

is given as

$$\phi(\boldsymbol{r}, t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(\boldsymbol{r}) = \sum_{n=0}^{\infty} a_n e^{\alpha_n t}\phi_n(\boldsymbol{r}) \tag{4}$$

The coefficients a_n can be determined from the initial condition (3) as

$$a_n = \frac{\int \phi(\boldsymbol{r})\phi_n(\boldsymbol{r})d\boldsymbol{r}}{\int \phi_n^2(\boldsymbol{r})d\boldsymbol{r}} \tag{5}$$

As is known, the α eigenvalue with the largest real part, α_0, is real, and the corresponding eigenfunction, the fundamental mode $\phi_0(\boldsymbol{r})$, is non-negative. The asymptotic behaviour is therefore

$$\phi(\boldsymbol{r}, t) \to a_0 e^{\alpha_0 t}\phi_0(\boldsymbol{r}) \tag{6}$$

and the other eigenvalues α_n describe, through (4), how fast the system reverts to the asymptotic state. It is also seen that the asymptotic behaviour is point kinetic, i.e. the space-time dependence of the flux is factorised into a time-dependent amplitude factor, and a shape function which is identical with the fundamental mode, i.e. the fundamental eigenfunction of the diffusion operator. As is also well known, for a critical system $\alpha_0 = 0$, i.e. the asymptotic solution is time-independent. For the case $\alpha_0 \neq 0$, no time-independent solutions exist.

Source-driven system

For the source-driven subcritical system, the equation corresponding to (1) is written as

$$\frac{1}{v}\frac{\partial \phi(\boldsymbol{r}, t)}{\partial t} = \boldsymbol{L} \cdot \phi(\boldsymbol{r}, t) + S(\boldsymbol{r}) \tag{7}$$

Here we only treat the case of a time-independent source, in which case (7) always has a time-independent static solution $\phi_s(\boldsymbol{r})$, obeying

$$\boldsymbol{L}\phi_s(\boldsymbol{r}) + S(\boldsymbol{r}) = 0 \tag{8}$$

To describe the relaxation of the system from an arbitrary initial condition to the static solution, one can still use the eigenfunction expansion method, although in a slightly modified way. The eigenfunctions to be used are still those of the operator $\boldsymbol{L}$. However, the static (asymptotic) solution $\phi_s(\boldsymbol{r})$ is no longer an eigenfunction of $\boldsymbol{L}$. An expansion of the flux in the form (4) is

therefore not practical in this case, since the asymptotic form would consist of the expansion of $\phi_s(\boldsymbol{r})$ into the eigenfunctions $\phi_n(\boldsymbol{r})$ and hence the transition to the asymptotic form would not be easily visible. It is therefore more practical to seek the time-dependent solution in the form

$$\phi(\boldsymbol{r}, t) = \phi_s(\boldsymbol{r}) + \varphi(\boldsymbol{r}, t) \tag{9}$$

From (7)-(9) it is seen that $\varphi(\boldsymbol{r}, t)$ satisfies the homogeneous (source-free) equation (1), hence its solution is worth to be given in the form of an eigenfunction expansion. This will yield the same result as (4), i.e.

$$\varphi(\boldsymbol{r}, t) = \sum_{n=0}^{\infty} a_n e^{\alpha_n t} \phi_n(\boldsymbol{r}) \tag{10}$$

with the only difference that the coefficients a_n are now given by

$$a_n = \frac{\int([\phi(\boldsymbol{r}) - \phi_s(\boldsymbol{r})])\phi_n(\boldsymbol{r})d\boldsymbol{r}}{\int \phi_n^2(\boldsymbol{r})d\boldsymbol{r}} \tag{11}$$

Hence the full time-dependent solution is given as

$$\phi(\boldsymbol{r}, t) = \phi_s(\boldsymbol{r}) + \sum_{n=0}^{\infty} a_n e^{\alpha_n t} \phi_n(\boldsymbol{r}) \tag{12}$$

Since the system is subcritical, $\Re e(\alpha_n) < 0, \ \forall n$, and the whole sum in the r.h.s. of (12) vanishes with increasing time. Eqn (12) shows the transition to the asymptotic state, and it is seen that this transition is not point kinetic. The speed of the initial flux shape to revert to the asymptotic depends on the eigenvalues α_n. In general, the deeper the system subcriticality, the faster the asymptotic state is reached. There is no similar statement for the source-free system, because there the speed of approaching the asymptotic state depends on the ratio between the real parts of the fundamental and higher order eigenvalues.

2.2 Neutron noise induced by cross section fluctuations

Source-free (critical) system

For this case now we select a concrete representation of the diffusion operator, corresponding to a bare homogeneous system. Hence, for a source-free system, the time-dependent equations read as

$$\frac{1}{v}\frac{\partial \phi(\boldsymbol{r}, t)}{\partial t} = D\nabla^2\phi + [(1-\beta)\nu\Sigma_f - \Sigma_a]\phi(\boldsymbol{r}, t) + \lambda C(\boldsymbol{r}, t), \tag{13}$$

$$\frac{\partial C(\boldsymbol{r}, t)}{\partial t} = \beta\nu\Sigma_f\phi(\boldsymbol{r}, t) - \lambda C(\boldsymbol{r}, t) \tag{14}$$

We will assume that without perturbations the system is critical, described by

$$\nabla^2\phi_0(\boldsymbol{r}) + B_0^2\phi_0(\boldsymbol{r}) = 0 \tag{15}$$

and

$$\beta\nu\Sigma_f\phi_0(\boldsymbol{r}) = \lambda C_0(\boldsymbol{r}) \tag{16}$$

with

$$B_0^2 = \frac{\nu\Sigma_f - \Sigma_a}{D} \tag{17}$$

and with the usual diffusion theory boundary condition

$$\phi(\boldsymbol{r}_B) = 0\ . \tag{18}$$

We will assume that the time-dependence of the flux is brought about by the space- and time-dependence of the absorption cross sections:

$$\Sigma_a \rightarrow \Sigma_a + \delta\Sigma_a(\boldsymbol{r}, t) \tag{19}$$

Since the small cross section fluctuations will induce small flux fluctuations and we shall use linearised equations, we write

$$\phi(\boldsymbol{r}, t) = \phi_0(\boldsymbol{r}) + \delta\phi(\boldsymbol{r}, t) \tag{20}$$

$$C(\boldsymbol{r}, t) = C_0(\boldsymbol{r}) + \delta C(\boldsymbol{r}, t) \tag{21}$$

Putting (19)-(21) into (13)-(14), subtracting the static equations, neglecting the second order term $\delta\Sigma_a(\boldsymbol{r}, t)\delta\phi(\boldsymbol{r}, t)$ and eliminating the fluctuations of the delayed neutrons by a temporal Fourier-transform, one obtains in the frequency domain the following equation for the neutron noise:

$$\nabla^2\delta\phi(\boldsymbol{r}, \omega) + B^2(\omega)\delta\phi(\boldsymbol{r}, \omega) = S(\boldsymbol{r}, \omega) \equiv \frac{\phi_0(\boldsymbol{r})}{D} \cdot \delta\Sigma_a(\boldsymbol{r}, \omega) \tag{22}$$

with $S(\boldsymbol{r}, \omega)$ representing the “noise source” whereas $\delta\Sigma_a(\boldsymbol{r}, \omega)$ is termed as the perturbation. Further,

$$B^2(\omega) = B_0^2 - \frac{\nu\Sigma_f}{DG_0(\omega)} \tag{23}$$

with

$$G_0(\omega) = \frac{1}{i\omega\left(\Lambda + \dfrac{\beta}{i\omega + \lambda}\right)} \tag{24}$$

being the zero-reactor transfer function of critical cores.

We shall now seek a solution of (22) with the eigenfunction expansion method as

$$\delta\phi(\boldsymbol{r}, \omega) = \sum_{n=0}^{\infty} a_n(\omega)\phi_n(\boldsymbol{r}) \tag{25}$$

where the eigenfunctions are now solutions to the equation

$$\nabla^2\phi_n(\boldsymbol{r}) + B_n^2\phi_0(\boldsymbol{r}) = 0 \tag{26}$$

with the same boundary conditions as (18). Obviously, criticality requires that B_0^2 in (26) is the same as in (17). Substitution of (25) into (22) yields, using the orthogonality of the eigenfunctions, an expression for the $a_n(\omega)$ in the form

$$a_n(\omega) = \frac{\int \delta\Sigma_a(\boldsymbol{r}, \omega)\phi_0(\boldsymbol{r})\phi_n(\boldsymbol{r})d\boldsymbol{r}}{D\left[-B_n^2 + B_0^2 - \dfrac{\nu\Sigma_f}{D \cdot G_0(\omega)}\right]\int \phi_n^2(\boldsymbol{r})d\boldsymbol{r}} \tag{27}$$

In writing out (25) with (27) it is practical to separate the first term, i.e. that for $n = 0$ from the rest:

$$\delta\phi(\boldsymbol{r}, \omega) = \delta\rho(\omega) \cdot G_0(\omega) \cdot \phi_0(\boldsymbol{r}) + \sum_{n=1}^{\infty} \frac{-\int \delta\Sigma_a(\boldsymbol{r}, \omega)\phi_0(\boldsymbol{r})\phi_n(\boldsymbol{r})d\boldsymbol{r}}{D\left[B_n^2 - B_0^2 + \dfrac{\nu\Sigma_f}{D \cdot G_0(\omega)}\right]\int \phi_n^2(\boldsymbol{r})d\boldsymbol{r}} \cdot \phi_n(\boldsymbol{r}) \tag{28}$$

where the reactivity of the perturbation, $\delta\rho(\omega)$, is given by

$$\delta\rho(\omega) = -\frac{1}{\nu\Sigma_f}\frac{\int \delta\Sigma_a(\boldsymbol{r}, \omega)\phi_0^2(\boldsymbol{r})d\boldsymbol{r}}{\int \phi_n^2(\boldsymbol{r})d\boldsymbol{r}} \tag{29}$$

From (28) it is seen that the first term becomes dominant in two distinct cases, so that the last term on the r.h.s. expressed by the sum can be neglected or is vanishing. Whenever this happens, the system behaviour becomes point kinetic, since, again, the time (frequency) and space dependence is factorised, and the space dependence is equal to that of the fundamental mode. The first case of point kinetic behaviour is that of the low frequencies, and this observation was made already by Weinberg and Schweinler [4]. As (24) shows, $G_0(\omega)$ diverges for $\omega \to 0$, whereas all terms in the sum on the r.h.s. of (28) remain finite in this limit. The only exception from this behaviour arises for perturbations whose reactivity effect is exactly zero, i. e. when $\delta\Sigma_a(\boldsymbol{r}, \omega)$ is orthogonal to $\phi_0^2(\boldsymbol{r})$. In that case point kinetic behaviour will never occur. The other possibility for point kinetic behaviour is when the perturbation is space-independent, i.e. when

$$\delta\Sigma_a(\boldsymbol{r}, \omega) = \delta\Sigma_a(\omega). \tag{30}$$

or, in other words, when the noise source is proportional to the static flux,

$$S(\boldsymbol{r}, \omega) = \frac{\phi_0(\boldsymbol{r})}{D} \cdot \delta\Sigma_a(\omega). \tag{31}$$

In that case, the whole sum in (28) is zero, i.e. $a_n = 0$ for all n values. Physically this means that the noise source, due to the perturbation (30), does not excite any of the higher order spatial modes, only the fundamental one, and hence point kinetic behaviour will prevail for any frequency.

Source-driven (subcritical) system

In order to distinguish that the system is now not critical we shall use slightly different notations. Eqn (13) is now replaced by

$$\frac{1}{v}\frac{\partial\phi(\boldsymbol{r}, t)}{\partial t} = D\nabla^2\phi + [(1-\beta)\nu\Sigma_f - \Sigma_a(\boldsymbol{r}, t)]\phi(\boldsymbol{r}, t) + \lambda C(\boldsymbol{r}, t) + S(\boldsymbol{r}) \tag{32}$$

and the static equation by

$$\nabla^2\phi_s(\boldsymbol{r}) + B_s^2\phi_s(\boldsymbol{r}) = S(\boldsymbol{r}) \tag{33}$$

The subscript "s" indicates the source-driven subcritical case, in order to distinguish from the geometrical buckling (fundamental eigenvalue) and the fundamental mode. In (33), B_s^2 stands for the static (non-critical) material buckling

$$B_s^2 = \frac{\nu\Sigma_f - \Sigma_a}{D} \tag{34}$$

The buckling corresponding to the fundamental mode $\phi_0(\boldsymbol{r})$ is related to the above as

$$B_0^2 = \frac{\nu\Sigma_f/k - \Sigma_a}{D} = B_s^2 - \frac{\rho_s\nu\Sigma_f}{D} \tag{35}$$

Here ρ_s is the static subcriticality of the source-driven system, not to be mixed up with the reactivity effect of the perturbation, $\delta\rho$.

To arrive to the noise equations, one again assumes that the time (frequency) dependence is induced by the fluctuations of the absorption cross sections, eqn (19). However the neutron noise needs to be defined as the deviation from the static subcritical flux, i.e. in contrast to (20) one writes

$$\phi(\boldsymbol{r}, t) = \phi_s(\boldsymbol{r}) + \delta\phi(\boldsymbol{r}, t) \tag{36}$$

Repeating now the same steps as the one leading to (22) will lead to a formally similar equation in the form

$$\nabla^2\delta\phi(\boldsymbol{r}, \omega) + B_s^2(\omega)\delta\phi(\boldsymbol{r}, \omega) = S(\boldsymbol{r}, \omega) \equiv \frac{\phi_s(\boldsymbol{r})}{D}\cdot\delta\Sigma_a(\boldsymbol{r}, \omega) \tag{37}$$

with

$$B_s^2(\omega) = B_s^2 - \frac{\nu\Sigma_f}{DG_0(\omega)} \tag{38}$$

One can now again seek the solution of (37) in the same form as in (25). After the same manipulations as before one arrives at the solution

$$\delta\phi(\boldsymbol{r},\omega) = \delta\rho(\omega)\cdot G_s(\omega)\cdot\phi_0(\boldsymbol{r}) + \sum_{n=1}^{\infty}\frac{-\int\delta\Sigma_a(\boldsymbol{r},\omega)\phi_s(\boldsymbol{r})\phi_n(\boldsymbol{r})d\boldsymbol{r}}{D\left[B_n^2 - B_0^2 + \dfrac{\nu\Sigma_f}{D\cdot G_s(\omega)}\right]\int\phi_n^2(\boldsymbol{r})d\boldsymbol{r}}\cdot\phi_n(\boldsymbol{r}) \qquad (39)$$

Here

$$G_s(\omega) = \frac{1}{i\omega\left(\Lambda + \dfrac{\beta}{i\omega+\lambda}\right) - \rho_s} \qquad (40)$$

is the zero reactor transfer function of subcritical cores [5], and the reactivity perturbation is given as

$$\delta\rho(\omega) = -\frac{1}{\nu\Sigma_f}\frac{\int\delta\Sigma_a(\boldsymbol{r},\omega)\phi_s(\boldsymbol{r})\phi_0(\boldsymbol{r})d\boldsymbol{r}}{\int\phi_0^2(\boldsymbol{r})d\boldsymbol{r}} \qquad (41)$$

The properties of the solution (39) are different from those of (28), therefore (39) is less suitable for the analysis of the asymptotic properties of the system. These differences are related to the differences in the physics of the two cases (critical and source-driven systems). First of all, in a source-driven system the point kinetic approximation means a space-time factorisation in which the space dependence is equal to that of the static flux $\phi_s(\boldsymbol{r})$. It is not possible to easily discern any asymptotics in (39) which would lead to such a behaviour. In particular, and in contrast to (28), the first term on the r.h.s. does not represent the point kinetic term. Consistently, this term does not become dominant either in any of the two cases which led to point kinetic behaviour in the critical system. What regards the case of low frequencies, as (40) shows, $G_s(\omega)$ remains finite even for $\omega = 0$, hence low frequencies do not induce the dominance of the first term. Likewise, since the static flux $\phi_s(\boldsymbol{r})$ is not orthogonal to any of the eigenmodes ϕ_n, a spatially constant perturbation will not lead to the vanishing of the sum in (39) since the static flux $\phi_s(\boldsymbol{r})$ is not orthogonal to any of the eigenmodes $\phi_n(\boldsymbol{r})$.

Based on the postulation of point kinetic behaviour in the form of

$$A(\omega)\phi_s(\boldsymbol{r}), \qquad (42)$$

on can try to use an expansion similar to (9) and (10) in the hope of extracting the conditions of point kinetic behaviour. That is, one writes

$$\delta\phi(\boldsymbol{r},\omega) = A(\omega)\phi_s(\boldsymbol{r}) + \sum_{n=0}^{\infty}a_n(\omega)\phi_n(\boldsymbol{r}) \qquad (43)$$

This trick is, however, not as effective in the present case as in Section 2.1 treating the asymptotic

behaviour in a steady system. Executing the same steps as before will lead to

$$-A(\omega)\left[\frac{\nu\Sigma_f}{D\cdot G_0(\omega)}\phi_s(\boldsymbol{r})+S(\boldsymbol{r})\right]+\sum_{n=0}^{\infty}a_n(\omega)[B_s^2(\omega)-B_n^2]\phi_n(\boldsymbol{r})\;=\;S(\boldsymbol{r},\omega) \tag{44}$$

For $\omega\to 0$ this can be simplified to

$$-A(\omega)S(\boldsymbol{r})+\sum_{n=0}^{\infty}a_n(\omega)[B_s^2(\omega)-B_n^2]\phi_n(\boldsymbol{r})\;=\;S(\boldsymbol{r},\omega) \tag{45}$$

From (45) it is seen that the frequency tending to zero is not sufficient alone to grant the validity of the point kinetic approximation. It is also necessary that the perturbation $S(\boldsymbol{r},\omega)$ has a special form. It can be shown that if the noise source can be factorised into a frequency dependent factor and a spatial shape equal to that of the static source, i.e.

$$S(\boldsymbol{r},\omega)\;=\;f(\omega)S(\boldsymbol{r}), \tag{46}$$

then

$$A(\omega)\;=\;-f(\omega) \tag{47}$$

and

$$a_n(\omega)\;=\;0;\;\forall n \tag{48}$$

is a solution of (45). That is, for vanishing frequencies, and a noise source that factorises as the static source, one will have

$$\delta\phi(\boldsymbol{r},\omega)\;=\;-f(\omega)\phi_s(\boldsymbol{r}) \tag{49}$$

This solution has already been derived in [6] with much less effort with the use of the Green's function technique. However, this solution is rather formal and useless, since, strictly speaking, it is only valid for $\omega=0$. It is at any rate not suitable to investigate the domain of validity of the point kinetic approximation with increasing frequencies, only for the demonstration of two facts. One is that low frequency alone is not sufficient to impose point kinetic behaviour on a source-driven system, which is a direct consequence of the physics. The second is the fact that the expansion of the solution of a source-driven system into spatial eigenfunctions of the diffusion or transport operator is rather ineffective in treating neutron noise problems, even in cases of low frequency or a special spatial shape of the perturbation. For the treatment of the reactor noise in source-driven systems the flux factorisation technique and the kinetic approximations are more useful.

3. FLUX FACTORISATION AND THE KINETIC APPROXIMATIONS

The possibility of using the Henry factorisation technique [1] for the definition and analysis of the kinetic approximations in source-driven systems was already investigated in [6]. Apparently only the case of fluctuations of the extraneous source were investigated there, but formally, the noise

source of the linearised equations, represented by the fluctuations of the absorption cross sections, play a formally identical role. Hence here we only summarize the main points of the analysis given in [6], by formulating it such that the perturbation can be either fluctuations of the extraneous source, or the cross section fluctuations. The system we investigate will be the same as in Section 2 above.

The reactor kinetic approximations are all based on a factorisation of the space-time dependent flux into an amplitude factor and a shape function as follows ([1], [2]). One writes

$$\phi(\boldsymbol{r}, t) = P(t)\psi(\boldsymbol{r}, t) \tag{50}$$

where $P(t)$ is the amplitude function and $\psi(\boldsymbol{r}, t)$ the shape function. The idea is that any change in reactor power should be represented by the amplitude factor, whereas deviations from the stationary flux shape be represented by the shape function $\psi(\boldsymbol{r}, t)$. To this order, and also to make the factorisation (50) unambiguous, one requires the normalisation condition

$$\frac{\partial}{\partial t}\int \psi(\boldsymbol{r}, t)\phi_0(\boldsymbol{r})d\boldsymbol{r} = 0 \tag{51}$$

where $\phi_0(\boldsymbol{r})$ is the fundamental mode, i.e. the solution of the eigenvalue equation

$$D\nabla^2\phi_0(\boldsymbol{r}) + \left(\frac{\nu\Sigma_f}{k_{eff}} - \Sigma_a\right)\phi_0(\boldsymbol{r}) = 0 \tag{52}$$

Actually, the choice of the weight function in the normalisation condition (51) is not crucial. Other weight functions are also possible. This question is discussed in [7].

We shall also assume that at $t = -\infty$, i.e. before the perturbation started, one had a stationary system with

$$\phi(\boldsymbol{r}, t = -\infty) = \phi_s(\boldsymbol{r}) \tag{53}$$

from which one has

$$P(t = -\infty) \equiv P_0 = 1 \tag{54}$$

and

$$\int \psi(\boldsymbol{r}, t)\phi_0(\boldsymbol{r})d\boldsymbol{r} = \int \phi_s(\boldsymbol{r})\phi_0(\boldsymbol{r})d\boldsymbol{r} \tag{55}$$

Equations (53) and (55) amount to the fact how the point kinetic approximation is defined, and it is consistent with the earlier definition in Section 2. Namely, deviations of the flux shape from that of the static (subcritical) flux will count as deviation from the point kinetics, whereas all changes of power that do not alter the shape of the static flux will count as point kinetic.

The usual way of developing the reactor physics approximations from (53) and (55) is to derive coupled equations for $P(t)$ and $\psi(\boldsymbol{r}, t)$. Here we shall only consider the linear case of small

perturbations. Then, similarly to the previous case, all time-dependent quantities will be split up to static (expected) values and fluctuations as

$$\phi(\boldsymbol{r}, t) = \phi_s(\boldsymbol{r}) + \delta\phi(\boldsymbol{r}, t) \tag{56}$$

$$P(t) = 1 + \delta P(t) \tag{57}$$

$$\psi(\boldsymbol{r}, t) = \phi_s(\boldsymbol{r}) + \delta\psi(\boldsymbol{r}, t) \tag{58}$$

The perturbation is represented by the fluctuations of the absorption cross section as before,

$$\Sigma_a(\boldsymbol{r}, t) = \Sigma_a + \delta\Sigma_a(\boldsymbol{r}, t) \tag{59}$$

Using (58) in (50) and neglecting the second order terms yields the following expressions in the time and frequency domain:

$$\begin{aligned} \delta\phi(\boldsymbol{r}, t) &= \phi_s(\boldsymbol{r}) \cdot \delta P(t) + \delta\psi(\boldsymbol{r}, t) \\ \delta\phi(\boldsymbol{r}, \omega) &= \phi_s(\boldsymbol{r}) \cdot \delta P(\omega) + \delta\psi(\boldsymbol{r}, \omega) \end{aligned} \tag{60}$$

To obtain equations for the fluctuations of the amplitude, $\delta P(\omega)$, and the shape function, $\delta\psi(\boldsymbol{r}, \omega)$, one has to substitute the flux factorisation (50) into the time-dependent equations, multiply by the fundamental mode $\phi_0(\boldsymbol{r})$ and integrate over the volume of the reactor. One also multiplies equation (52) by $\psi(\boldsymbol{r}, t)$, integrates, and subtracts the two equations. This manipulation will lead to the point kinetic equations for $P(t)$ in the form

$$\begin{cases} \dfrac{d\delta P(t)}{dt} = \dfrac{\rho - \beta}{\Lambda}\delta P(t) + \lambda\delta C(t) + \delta\rho(t) \\ \dfrac{d\delta C(t)}{dt} = \dfrac{\beta}{\Lambda}\delta P(t) - \lambda\delta C(t) \end{cases} \tag{61}$$

Here, the following functions have been introduced:

$$\delta C(t) \equiv \frac{\int \delta C(\boldsymbol{r}, t)\phi_0(\boldsymbol{r})d\boldsymbol{r}}{\frac{1}{\nu}\int \phi_0^2(\boldsymbol{r})d\boldsymbol{r}} \tag{62}$$

and

$$\delta\rho(t) \equiv -\frac{\int S(\boldsymbol{r}, t)\phi_0(\boldsymbol{r})d\boldsymbol{r}}{\nu\Sigma_f\int \phi_s(\boldsymbol{r})\phi_0^+(\boldsymbol{r})d\boldsymbol{r}} \tag{63}$$

and $S(\boldsymbol{r}, t)$ is the same as in (37).

The point kinetic equations can be solved by direct temporal Fourier transform of Eqn. (61). Eliminating the delayed neutron precursors leads to the frequency domain solution

$$P(\omega) = G_\rho(\omega)\delta\rho(\omega) \tag{64}$$

where $G_\rho(\omega)$ is the zero-reactor transfer function of the subcritical system, given by (40).

The equation for the shape function is more involved, and we shall here only treat two simple cases, i.e. the determination of $\psi(\boldsymbol{r}, t)$ in the point kinetic and adiabatic approximations. The point kinetic approximation actually means to assume

$$\psi(\boldsymbol{r}, t) = \phi_s(\boldsymbol{r}) \tag{65}$$

for all time instants. Thus the space-time dependent flux is given as

$$\delta\phi(\boldsymbol{r}, \omega) = \phi_s(\boldsymbol{r}) \cdot \delta P(\omega) = G_\rho(\omega)\delta\rho(\omega)\phi_s(\boldsymbol{r}) \tag{66}$$

According to this definition the reactor behaves in a point-kinetic manner as long as the flux shape does not deviate from that of the static flux of the subcritical, source-driven reactor.

The solution in the point kinetic approximation given by (66) is defined for all frequencies. Hence the validity of this approximation can be quantitatively checked against the solution of the full space-frequency dependent equation (37). Such a comparison has been performed for a few specific cases of perturbations in [6]. In accordance with what has been stated in the foregoing, the point kinetic approximation performs well for noise sources having the same shape as the static source, but it breaks down even for low frequencies for other perturbations.

The definition of the adiabatic approximation for source-driven systems is different from what one would intuitively suggest. Since the system is now subcritical, the static equation (33) has always a solution $\phi(\boldsymbol{r}, t)$ with a time-dependent source $S(\boldsymbol{r}, t)$ at any time instant t such that t is only a parameter and not a variable. It is tempting to define the adiabatic approximation as determining $\phi(\boldsymbol{r}, t)$ from such a simple calculation.

However, as is described in [6], such a solution is very poor because in treating a fully static equation, one neglects the contribution from the delayed neutrons completely. At low frequencies this solution would be exact, but with increasing frequencies ($\omega > \lambda$) the decrease of the amplitude due to the disappearing of the delayed neutrons from the dynamic response would not be accounted for. Thus it is much more efficient to still use the factorisation (50), determine the amplitude factor $P(t)$ from the point kinetic equations, and determine the shape function $\psi(\boldsymbol{r}, t)$ from a static equation by using the normalisation (51). This normalisation is not as trivial as in the case of critical systems, where the shape function has to be determined from an eigenvalue equation and needs to be normalised anyway. By the above described strategy the delayed neutrons (or their absence) are accounted for, even if only with a simple space dependence.

Thus, for the definition of the adiabatic approximation, we re-write the time-dependent equation as

$$\frac{1}{v}\frac{\partial\phi(\boldsymbol{r}, t)}{\partial t} = D\nabla^2\phi(\boldsymbol{r}, t) + [\nu\Sigma_f - \Sigma_a(\boldsymbol{r}, t)]\phi(\boldsymbol{r}, t) - \frac{\partial C(\boldsymbol{r}, t)}{\partial t} + S(\boldsymbol{r}) \tag{67}$$

Introducing the factorisation (50) into (67) and neglecting all time derivatives leads to the equation

$$D\nabla^2\psi_{ad}(\boldsymbol{r}, t) + [\nu\Sigma_f - \Sigma_a(\boldsymbol{r}, t)]\psi_{ad}(\boldsymbol{r}, t) + \frac{S(\boldsymbol{r})}{P(t)} = 0 \tag{68}$$

Actually, eqn (68) can be further simplified. According to (51) or (55), the shape function needs to be properly normalised. If we had not neglected time derivatives when going over from (67) to (68), this would have been granted. However, due to the neglections of the time derivatives, the solution of (68) will not, in general, be properly normalized. Since the solution of (68) depends linearly on the last term (which is the inhomogeneous term in the equation), we can replace the factor $1/P(t)$ with unity, since the normalisation will overrule the effect of it anyway. Thus the adiabatic equation for the shape function will be

$$D\nabla^2\psi_{ad}(\boldsymbol{r}, t) + [\nu\Sigma_f - \Sigma_a(\boldsymbol{r}, t)]\psi_{ad}(\boldsymbol{r}, t) + S(\boldsymbol{r}) = 0 \tag{69}$$

with the further condition that $\psi_{ad}(\boldsymbol{r}, t)$ must fulfil the normalisation condition (55). How the normalisation is achieved is described in [6]. Having found $\psi_{ad}(\boldsymbol{r}, t)$, the fluctuation of the flux shape in the adiabatic approximation is given as

$$\delta\psi_{ad}(\boldsymbol{r}, t) = \psi_{ad}(\boldsymbol{r}, t) - \phi_s(\boldsymbol{r}) \tag{70}$$

The solution for the noise in the adiabatic approximation is given as

$$\delta\phi_{ad}(\boldsymbol{r}, \omega) = \phi_s(\boldsymbol{r}) \cdot \delta P(\omega) + \delta\psi_{ad}(\boldsymbol{r}, \omega) \tag{71}$$

The domain of applicability of this approximation was also investigated both analytically and quantitatively for some basic noise source types in [6]. It was found that for the cases when the noise source has the same space dependence as the static source, i.e. for cases defined by (46), the point kinetic approximation becomes exact at low frequencies, but the adiabatic approximation does not give any contributions. For perturbations that do not fulfil (46), the point kinetic approximation breaks down, but the adiabatic approximation works fairly well and becomes exact at low frequencies. Several quantitative examples are given in [6] with illustrations to support these statements.

4. CONCLUSIONS

Investigation of a few simple cases of time- or frequency dependent behaviour in source-free and source-driven systems, both in steady and perturbed systems, gave some insight into the mathematical and physical differences between the two type of systems. The applicability of the eigenfunction expansion and the flux factorisation methods was touched upon, and the validity of the point kinetic and the adiabatic approximations was investigated. It was found, not surprisingly, that the different physical properties of the two type of systems are reflected in the applicability of the mathematical methods and the validity of the kinetic approximations. These investigations contribute to the understanding of the dynamic behaviour of source-driven subcritical systems.

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