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THE $\mathrm{LTS_N}$ EXPONENTIAL NODAL METHOD FOR ONE-SPEED $\mathbf{X},\mathbf{Y}-$ GEOMETRY DISCRETE ORDINATES PROBLEMS IN HETEROGENEOUS MEDIA

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ABSTRACT

We describe an exponential LTS_N nodal method applied to $X,Y-$ geometry heterogeneous neutron transport problems in the discrete ordinates S_N formulation. We approximate the transverse leakage terms by exponential functions with prescribed spatial decay constants that depend on the material properties of the region which the particles leave behind as they stream across the heterogeneous system. We show in numerical experiments that LTS_N-exponential nodal method $(LTS_N - ExpN)$ generates very accurate results when compared to the conventional transport nodal methods for coarse-mesh shielding S_N problems, specially in highly absorbing media.

Key Words: exponential nodal method, shielding problems, discrete ordinates.

1. INTRODUCTION

The present Laplace Transform S_N- exponential nodal $(LTS_N - ExpN)$ method is based on the essence of the spectral nodal methods for discrete ordinates problems [1], wherein the only approximation involved is the approximation for the transverse leakage terms. Therefore, the scattering source terms in the one-dimensional transverse-integrated S_N nodal equations are treated exactly in the offered $LTS_N - ExpN$ method. In this method, we approximate the transverse leakage terms by exponential functions, that are chosen based on the physics of shielding problems, where the neutron flux attenuates exponentially with increasing distance from the source. By considering partial anisotropy in the exponential approximation, we assume that the neutron fluxes decrease exponentially along the region interfaces with attenuation constants that depend upon the nuclear properties of the region which the neutrons leave behind, as they stream across the heterogeneous shielding structure.

An outline of the remainder of this paper follows. In Sec.2, we describe the $LTS_N - ExpN$ methodology pointing out the mathematical preliminaries and the essence of the offered $LTS_N - ExpN$ algorithm. In Sec. 3, numerical results to a classical shielding test problem are given, and a brief discussion with suggestions for future work are offered in Sec. 4.

2. METHODOLOGY

Let us consider a rectangular region $\Re$ of width a and height b inside the domain, where the material parameters are constant. Our first step is to transverse-integrate the S_N equations inside region $\Re$. The results are

$$\sum_{i=1}^{I-1}\left[\frac{d\Psi_{my}^{(i)}}{dy}(y)+\frac{\sigma_{t_i}}{\eta_m}\Psi_{my}^{(i)}(y)-\frac{\sigma_{s_i}}{4\eta_m}\sum_{n=1}^{M}w_n\Psi_{ny}^{(i)}(y)\right]=\sum_{i=1}^{I-1}S_{my}^{(i)}(y)\,, \tag{1}$$

where we have defined the source term as

$$S_{my}^{(i)}(y)=\frac{1}{a\eta_m}\left[Q_y^{(i)}(y)-\mu_m\left[\Psi_m(x_{i+1},y)-\Psi_m(x_i,y)\right]\right]\;, \tag{2}$$

and

$$\sum_{j=1}^{J-1}\left[\frac{d\Psi_{mx}^{(j)}}{dx}(x)+\frac{\sigma_{t_j}}{\eta_m}\Psi_{mx}^{(j)}(x)-\frac{\sigma_{s_j}}{4\eta_m}\sum_{n=1}^{M}w_n\Psi_{nx}^{(j)}(x)\right]=\sum_{j=1}^{J-1}S_{mx}^{(j)}(x)\,, \tag{3}$$

with similar definition

$$S_{mx}^{(j)}(x)=\frac{1}{b\mu_m}\left[Q_x^{(j)}(x)-\eta_m\left[\Psi_m(x,y_{j+1})-\Psi_m(x,y_j)\right]\right]\;. \tag{4}$$

Here the notation is standard, with I and J being the number of regions in the x and y directions respectively, and $M=N(N+2)/2$ for the $X,Y-$ geometry S_N formulations. We remark that the transverse leakage terms are included in Eqs. (5) and (6) as

$$\frac{\mu_m}{a}\left[\Psi_m(x_{i+1},y)-\Psi_m(x_i,y)\right] \tag{5}$$

and

$$\frac{\eta_m}{b}\left[\Psi_m(x,y_{j+1})-\Psi_m(x,y_j)\right]\;. \tag{6}$$

Now we apply the Laplace transform to Eq.(1) with respect to the y variable. The result in matrix notation appears as

$$(sI-A_y^{(i)})\overline{\Psi}_y^{(i)}(s)=\Psi_y(0)+\overline{S}_y^{(i)}(s)\,. \tag{7}$$

Similar result is obtained by applying the Laplace transform to Eq.(3) with respect to the x variable.

Therefore, from Eq.(7), we obtain

$$\overline{\Psi}_y^{(i)}(s) = (sI - A_y^{(i)})^{-1}[\Psi_y(0) + \overline{S}_y^{(i)}(s)] \ . \tag{8}$$

Furthermore, we apply the Laplace inverse transformation to Eq.(8), and we write

$$\Psi_y^{(i)}(y) = \pounds^{-1}\{(sI - A_y^{(i)})^{-1}[\Psi_y(0) + \overline{S}_y^{(i)}(s)]\} \ . \tag{9}$$

In order to determine $\pounds^{-1}\{(sI - A_y^{(i)})^{-1}\}$, we use the diagonalizability of matrix $A_y^{(i)}$ to write

$$\pounds^{-1}\{(sI - A_y^{(i)})^{-1}\} = \mathbf{V}_y^{(i)}\pounds^{-1}\{(sI - \mathbf{D}_y^{(i)})^{-1}\}(\mathbf{V}_y^{(i)})^{-1} = \mathbf{V}_y^{(i)}e^{\mathbf{D}_y^{(i)}y}(\mathbf{V}_y^{(i)})^{-1} \, , \tag{10}$$

that we substitute into Eq. (9) and we obtain

$$\Psi_y^{(i)}(y) = [\mathbf{V}_y^{(i)}e^{\mathbf{D}_y^{(i)}y}(\mathbf{V}_y^{(i)})^{-1}]\Psi_y(0) + [\mathbf{V}_y^{(i)}e^{\mathbf{D}_y^{(i)}y}(\mathbf{V}_y^{(i)})^{-1}] * S_y^{(i)}(y) \, , \tag{11}$$

where the notation $*$ stands for the convolution operation.

At this point, based on the physics of shielding problems, we assume that:
(i) the neutron flux attenuates exponentially with increasing distance from the source along the edges of each region inside the domain;
(ii) the attenuation constant depends upon the nuclear data of the region the neutrons leave behind as they stream across the system.
In this paper, we identify the attenuation constant as the macroscopic absorption cross section of the region that the neutrons leave behind.
With this heuristic approximation [5], we claim that for diffusive regions, where the macroscopic absorption cross sections are small, the attenuation of the transverse leakage term (5) along the y direction is smoother than for highly absorbing regions, where the absorption event dominates.
Similar reasoning applies for the transverse leakage term (6) along the x direction.

3. NUMERICAL RESULTS

To model the test problem, we used the level symmetric LQ_N quadrature set with $N = 4$ [2]. The model problem consists of a uniform isotropic neutron source $(Q = 1)$ surrounded by a shielding material $(Q = 0)$, viz Fig.1. This classical shielding problem was considered in Ref.[3].

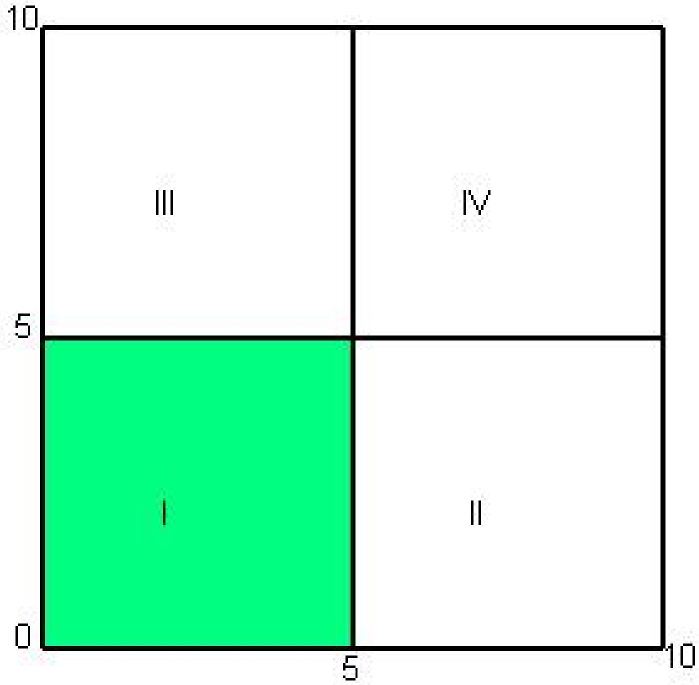

Figure 1. Model problem

Reflective boundary conditions apply at $x = 0$ and $y = 0$, and vacuum boundary conditions apply at the boundaries $x = 10$ and $y = 10$. The cross sections and source intensities are displayed in Table I.

Table I. Cross sections and source intensity Q

Region	Q	σ_t	σ_s
I	1	1	0.5
II	0	2	0.1
III	0	2	0.1
IV	0	2	0.1

Table II shows the average scalar fluxes in regions I, II and IV generated by the conventional LN method [3], two distinct spectral nodal methods, namely the $SGF - CN$ method [4] and the $SGF - ExpN$ method [5] on various spatial grids, and the present $LTS_N - ExpN$ method.

As we see, the maximum relative deviation generated by the offered $LTS_N - ExpN$ method with respect to the results generated by its companion $SGF - ExpN$ method on a spatial grid composed of 30 nodes in each spatial direction, was 3.70% for the average scalar flux in region II.

Table II. Numerical Results (S_4- quadrature set)

Numerical Method (spatial grid)	Average Scalar Flux Region I	Region II	Region IV
LN (Ref. [3])			
10×10	0.1676×10^1	0.4170×10^{-1}	0.1986×10^{-2}
20×20	0.1676×10^1	0.4161×10^{-1}	0.1990×10^{-2}
30×30	0.1676×10^1	0.4159×10^{-1}	0.1992×10^{-2}
$SGF - CN$ (Ref. [4])			
10×10	0.1676×10^1	0.4290×10^{-1}	0.2850×10^{-2}
20×20	0.1676×10^1	0.4165×10^{-1}	0.2019×10^{-2}
30×30	0.1676×10^1	0.4163×10^{-1}	0.2000×10^{-2}
$SGF - ExpN$ (Ref. [5])			
10×10	0.1676×10^1	0.4169×10^{-1}	0.2000×10^{-2}
20×20	0.1676×10^1	0.4161×10^{-1}	0.1995×10^{-2}
30×30	0.1676×10^1	0.4161×10^{-1}	0.1993×10^{-2}
$LTS_N - ExpN$	0.1673×10^1	0.4321×10^{-1}	0.1999×10^{-2}
Relative Deviation*	0.18%	3.70%	0.30%

$\star$ with respect to the 30×30 $SGF - ExpN$ results.

4. DISCUSSION

We described an analytical numerical method, that we refer to as the LTS_N method for monoenergetic S_N fixed source problems in X,Y-geometry. In this method, the only approximation involved is in the transverse leakage terms of the transverse-integrated S_N equations within each homogenized region of the heterogeneous domain. The scattering source terms are treated analytically. Therefore, in slab-geometry S_N problems the LTS_N method generates numerical solutions that are completely free from spatial truncation errors.

Based on the physics of radiation shielding problems, where the neutron flux attenuates exponentially with increasing distance from the source, we approximate the transverse leakage terms by exponential functions with prescribed attenuation constants. Therefore, by considering partial anisotropy in the exponential approximation, we apply the Laplace Transform to the transverse-integrated S_N equations within a given homogenized region and use continuity conditions to solve the problem in heterogeneous domains. As with the partial anisotropy, in the present $(LTS_N - ExpN)$ method, we consider that the neutron fluxes decrease exponentially along the region-edges with attenuation constant that is defined as the macroscopic absorption cross section of the region which the neutrons leave behind, as they stream across the domain.

It is well known that one serious drawback of coarse-mesh methods is the reduced resolution in the sense that accurate point values of the neutron flux, or other related quantities, are somewhat difficult to obtain from the coarse-mesh solution. Therefore, we suggest, as a future work, to develop approximate reconstruction schemes in order to recover the spatial profile of the neutron flux.

In conclusion, even though modern computers and the ever increasing power and capacity of computational platforms have lead the community as a whole to move farther from simplifying assumptions, the limits of accuracy of the offered $(LTS_N - ExpN)$method need to be investigated more deeply, since the quality of the results is degraded, as seen by the results displayed in Table II. Moreover, we intend to work on generating the benchmark results for the given test problem numerically on a set of fine grids, and use Richardson extrapolation technique [6] to obtain spatially converged solution. The implementation and testing of these ideas must await future work.

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