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SPECTRAL GREEN'S FUNCTION LINEAR-NODAL METHOD FOR MONOENERGETIC X,Y-GEOMETRY EXTERNAL SOURCE DISCRETE ORDINATES PROBLEMS

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ABSTRACT

A new spectral nodal method is developed for the solution of one-speed discrete ordinates (S_N) problems with isotropic scattering in X,Y-geometry. In this method, the spectral Green's function (SGF) scheme, originally developed for solving S_N problems in slab geometry with no spatial truncation error, is generalized to solve the one-dimensional transverse-integrated S_N nodal equations wherein we consider linear polynomial approximation for the transverse leakage terms. To solve the resulting SGF-linear nodal (SGF-LN) equations we implement the full-node inversion (FNI) iterative scheme, which uses the best available estimates for the node-entering quantities to evaluate the node angular quantities in *all* the exiting directions as the equations are swept across the system. We give numerical results that illustrate the accuracy of the SGF-LN method for coarse-mesh calculations.

Key Words: neutral particle transport, discrete ordinates, spectral nodal method

1. INTRODUCTION

We develop a new nodal method for one-speed discrete ordinates (S_N) deep penetration problems in X,Y geometry by following the essence of the spectral nodal methods, that solve, without spatial truncation errors, the one-dimensional transverse-integrated S_N nodal equations, *only* approximating the transverse-leakage terms; the scattering source terms are treated analytically. This is in contrast to the conventional weighted diamond-difference form of nodal methods, that are based on polynomial approximations for both the transverse leakage and the scattering source terms [1]. To solve the transverse-integrated S_N nodal equations in our method, we follow the steps taken in the spectral Green's function - constant nodal (SGF-CN) method [2] and in the SGF-exponential nodal (SGF-ExpN) method [3] for one-speed X,Y-geometry S_N problems. We approximate the transverse leakages through the edges of each spatial node by linear polynomial functions, so we call our method the SGF - linear nodal (SGF-LN) method. Therefore, in contrast to the previous SGF-CN and the SGF-ExpN methods, the offered SGF-LN method is based on two node transverse - integrations of the S_N equations for each spatial variable. This

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procedure yields the transverse-integrated S_N nodal equations for the zero'th-order spatial moments of the angular flux, and the transverse-integrated S_N nodal equations for the first-order spatial moments of the angular flux [4]. To obtain iterative solutions of the discretized SGF-LN equations, we use the full node inversion (FNI) scheme. This iterative scheme uses the most recent available estimates for the incoming node-edge spatial moments of the angular fluxes to calculate such quantities in all exiting directions as the equations are swept across the spatial grid set up on the rectangular domain.

2. METHODOLOGY

Let us consider a spatial grid on the rectangular domain, where each spatial node $D_{i,j}$ has width h_i and height k_j, i = 1 : I, j = 1 : J. Now we define the generalized transverse-integration operator

$$L_u^{(\ell)} = \frac{(2\ell+1)}{r_s}\int_{u_{s-1/2}}^{u_{s+1/2}} P_\ell\left[\frac{2(u-u_s)}{r_s}\right]\bullet \; du \quad , \tag{1}$$

where u = x or y, s = i or j, r = h or k, $\ell = 0$ or 1 respectively, and the midpoint

$$u_s = \frac{u_{s+1/2} + u_{s-1/2}}{2} \quad , \tag{2}$$

with P_ℓ being defined as the ℓ'th degree Legendre polynomial.

First we apply $L_y^{(0)}$ and $L_x^{(0)}$ to the one-speed X,Y-geometry S_N equations to obtain the one-dimensional transverse-integrated S_N nodal equations for the zero'th - order spatial moments of the angular flux for the x direction and the y direction respectively. Then we apply $L_y^{(1)}$ and $L_x^{(1)}$ to the S_N equations and we obtain the one-dimensional transverse integrated S_N nodal equations for the first-order spatial moments of the angular flux for the x and the y directions respectively.

Now we assume that the transverse leakage terms are linear polynomial functions along the edges of each node $D_{i,j}$. By substituting these linear leakage approximations into the one-dimensional transverse integrated S_N nodal equations and further introducing linear decoupling approximations, as in the conventional LN method [4], we may write the transverse integrated S_N linear nodal equations in the following generalized form

$$\frac{\Lambda_m}{\sigma_{T_{i,j}}}\frac{d}{du}F_m^{(k)}(u) + F_m^{(k)}(u) = \frac{c_{i,j}}{4}\sum_{n=1}^{M}F_n^{(k)}(u)\omega_n + a_{m,u}^{(k)}\frac{2(u-u_s)}{r_s} + b_{m,u}^{(k)} \quad , \tag{3}$$

$$u \in D_{i,j} \; , \; m = 1:M \; .$$

Moreover we write the generalized form of the local general solution of Eq. (3)

$$F_m^{(k)}(u) = \sum_{\ell=1}^{M} C_\ell F_{m,\ell}^h(u) + F_m^{p,(k)}(u) \qquad (4)$$
$$u \in D_{i,j} \ , \ k = 0 \text{ or } 1 \ , \ m = 1:M \ .$$

Here, C_ℓ is an arbitrary constant, $F_{m,\ell}^h(u)$ is an elementary solution of the homogeneous ordinary differential equations associated with Eq. (3) and $F_m^{p,(k)}(u)$ is the particular solution. We remark that the homogeneous solution is independent of the value of k. However, the particular solution depends upon k, as the coefficients *a* and *b* in Eq. (3) do depend upon the value of k. In order to determine the homogeneous solution, we use the spectral analysis described in Ref. [2] In addition, in order to determine the expression for the particular solution of Eq. (3), we assume that $F_m^{p,(k)}(u)$ in Eq. (4) is a linear polynomial function of u, substitute it into Eq. (3), and then we determine the coefficients. Following this procedure, we are able to solve Eq. (3) analytically inside each node $D_{i,j}$ of the spatial grid set up on the domain.

Furthermore, we derive the discretized SGF-LN equations for one spatial node, that are formed by the standard 3M zero'th- and first-order spatial moment discretized S_N balance equations, as described in Ref. [4], together with the non-standard 4M SGF auxiliary equations

$$\overline{F}_{m,i,j}^{(k)} = \sum_{\Lambda_n > 0} \lambda_{m,n} F_{m,s-1/2}^{(k)} + \sum_{\Lambda_n < 0} \lambda_{m,n} F_{m,s+1/2}^{(k)} + \beta_m^{(k)} \qquad (5)$$
$$m = 1:M \ , \ k = 0 \text{ or } 1 \ , \ s = i \text{ or } j \ .$$

Since the transverse-integrated S_N linear nodal Eq. (3) has the same general form for u = x or y and k = 0 or 1, we use the SGF auxiliary Eq. (5) for the x direction (s = i) with k = 0 and 1, and for the y direction (s = j) with k = 0 and 1. Therefore, in the 4M SGF auxiliary Eq. (5), $\lambda_{m,n}$ and $\beta_m^{(k)}$, k = 0 or 1, are determined by requiring that the general solution of Eq. (3), given by Eq. (4), has node-average and node-edge average angular quantities that for all constants C_ℓ satisfy the appropriate SGF auxiliary equation. Once we have determined $\lambda_{m,n}$ and $\beta_m^{(k)}$ for each node $D_{i,j}$ the 3M zero'th- and first-order spatial moment discretized S_N balance equations, together with the 4M SGF auxiliary equations constitute the discretized SGF-LN equations. Using appropriate boundary and continuity conditions, we perform full node inversions (FNI) to solve the SGF-LN equations on systems consisting of many nodes. That is, the FNI iterative scheme uses the most recent available estimates for the incoming node-edge average angular quantities to calculate the exiting node-edge average quantities in *all* the angular directions.

3. NUMERICAL RESULTS

To model the test-problem we present in this section, we used the level-symmetric LQ_N quadrature set with N = 4 [5] and a convergence criterion requiring the discrete maximum norm of the relative deviation between two iterates to be $\leq 10^{-5}$. The model problem consists of a uniform isotropic neutron source

(Q = 1) surrounded by a shielding material (Q = 0), viz Fig. 1. This shielding problem was considered in Ref. [6].

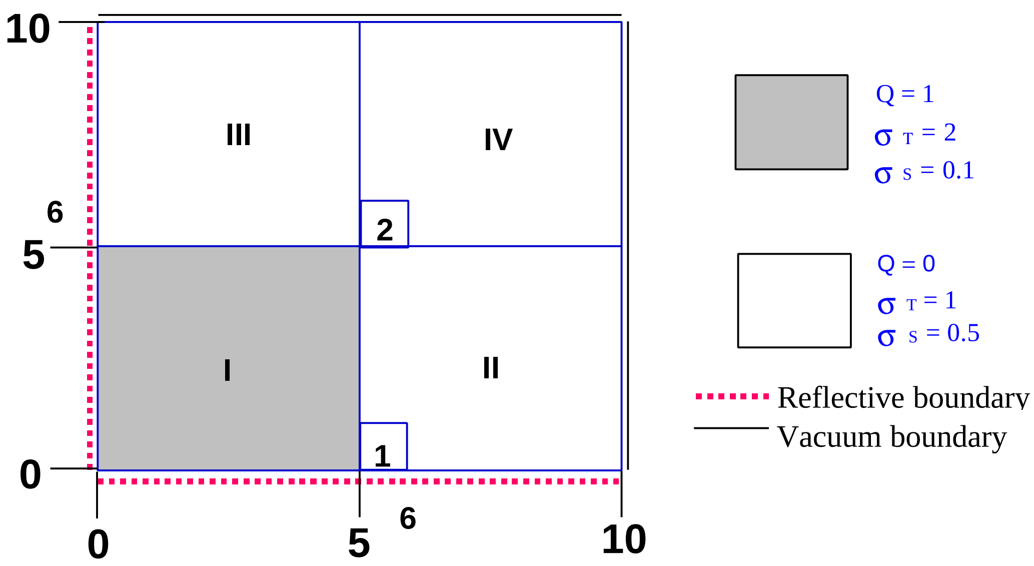

Fig. 1. Model Problem

Table I shows the average scalar fluxes in regions I, II, IV and in the localized regions 1 and 2 generated by the conventional LN method [4,6] and three distinct spectral nodal methods, namely the SGF-CN method [2], the SGF-ExpN method [3] and the present SGF-LN method on various spatial grids. The numerical results listed in bold face in Table I are referred to as valid results in this numerical experiment, since the relative deviations with respect to the fine-grid (100×100) LN results are less than 1%. Moreover, we note that the SGF-LN code generated valid results with a CPU running time that was 42% shorter than the execution times of the LN and the SGF-ExpN codes to generate valid results. This is also the case for the SGF-CN code, in which case the SGF-LN code generated valid results with a CPU running time that was 78% shorter.

In conclusion, we remark that the tedious algebra involved in the derivation of the SGF-LN equations and in setting them in the appropriate form for the computational implementation of the FNI iterative scheme has been overwhelmed by the efficiency of the SGF-LN code that generates valid coarse-mesh results in shorter CPU time. This is due to the fact that the SGF-LN method achieves comparable accuracies on spatial grids coarser than those required by the other nodal methods referenced in this summary.

Table I. Numerical Results for the Model-Problem

SPATIAL GRID	REGION I	REGION II	REGION IV	REGION 1	REGION 2
LN METHOD - Ref. [6]					
10×10 [b]	**1.676** [d] **(0.00)**	**0.4170E-1**[a] **(0.26)**[c]	**0.1986E-2 (0.30)**	**0.2151 (0.33)**	0.4148E-1 (3.62)
20×20	**1.676 (0.00)**	**0.4160E-1 (0.02)**	**0.1990E-2 (0.00)**	**0.2145 (0.00)**	**0.4006E-1 (0.07)**
40×40	**1.676 (0.00)**	**0.4159E-1 (0.00)**	**0.1992E-2 (0.00)**	**0.2144 (0.00)**	**0.4003E-1 (0.03)**
100×100	**1.676**	**0.4159E-1**	**0.1992E-2**	**0.2144**	**0.4004E-1**
SGF-CN METHOD - Ref. [3]					
10×10	**1.676 (0.00)**	0.4290E-1 (3.14)	0.2850E-2 (43.07)	0.2371 (10.58)	0.4648E-1 (16.11)
20×20	**1.676 (0.00)**	**0.4165E-1 (0.14)**	0.2019E-2 (1.33)	**0.2146 (0.00)**	**0.4013E-1 (0.24)**
40×40	**1.676 (0.00)**	**0.4163E-1 (0.09)**	**0.2000E-2 (0.40)**	**0.2145 (0.00)**	**0.4006E-1 (0.07)**
SGF-ExpN METHOD - Ref. [3]					
10×10	**1.676 (0.00)**	**0.4169E-1 (0.24)**	**0.2000E-2 (0.40)**	**0.2149 (0.23)**	0.4140E-1 (3.42)
20×20	**1.676 (0.00)**	**0.4161E-1 (0.04)**	**0.1995E-2 (0.00)**	**0.2145 (0.00)**	**0.4007E-1 (0.09)**
40×40	**1.676 (0.00)**	**0.4161E-1 (0.04)**	**0.1993E-2 (0.00)**	**0.2144 (0.00)**	**0.4004E-1 (0.02)**
SGF-LN METHOD					
10×10	**1.676 (0.00)**	**0.4163E-1 (0.10)**	**0.1991E-2 (0.05)**	**0.2144 (0.00)**	**0.4021E-1 (0.42)**
20×20	**1.676 (0.00)**	**0.4160E-1 (0.02)**	**0.1992E-2 (0.00)**	**0.2144 (0.00)**	**0.4005E-1 (0.02)**
40×40	**1.676 (0.00)**	**0.4159E-1 (0.00)**	**0.1992E-2 (0.00)**	**0.2144 (0.00)**	**0.4004E-1 (0.00)**
100×100	**1.676 (0.00)**	**0.4159E-1 (0.00)**	**0.1992E-2 (0.00)**	**0.2144 (0.00)**	**0.4004E-1 (0.00)**

a. Read as 0.4170×10^{-1}
b. Number of nodes in x-direction × number of nodes in y-direction
c. Relative deviation (%) with respect to the fine-grid (100×100) LN results
d. bold face for the relative deviations less than 1%.

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