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# SPECTRAL ELEMENT METHOD FOR THE NEUTRON DIFFUSION EQUATION ON A TRIANGULAR MESH

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## ABSTRACT

A Spectral Element Method to approximate the Lambda Modes problem for reactors with hexagonal geometry has been developed. This method is based on the expansion of the neutron flux in terms of the modified Dubiner's polynomials on a triangular mesh. This mesh is fixed and the accuracy of the method is improved increasing the degree of the polynomials expansions without the necessity of remeshing. The performance of method has been tested obtaining the dominant Lambda modes of different 2D reactor benchmark problems.

*Key Words:* Lambda Modes, Spectral Element Methods, VVER Reactors, Hexagonal geometry

## 1. INTRODUCTION

For a given configuration of a nuclear reactor core it is always possible to force its criticality dividing the neutron production rate due to fission by a positive number, $\lambda$, obtaining a neutron balance equation of the form

$$\mathcal{L}\Phi = \frac{1}{\lambda}\mathcal{M}\Phi, \tag{1}$$

where $\mathcal{L}$ is the neutron loss operator and $\mathcal{M}$ is the neutron production operator. Equation (1) is known as the Lambda modes equation. The largest eigenvalue, $\lambda$, satisfying (1) is the k-effective ($k_{\text{eff}}$) of the reactor core and the corresponding eigenfunction describes the neutron flux distribution for a stationary configuration of the core [1].

The Lambda modes problem has been studied for reactors with rectangular geometry as the PWR and BWR [2, 3]. To discretize the problem for reactors with a hexagonal geometry, as for example the VVER reactors, a different strategy is necessary because the design of these reactors defines a natural mesh with hexagonal cells instead of rectangular cells.

Different methods have been proposed to solve the neutron diffusion equation on hexagonal geometry as, for example, the Fourier transform method [4], the conformal mapping method [5],

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the polynomial expansion nodal method [6], etc. All these methods compute the $k_{\text{eff}}$ and the stationary neutron flux in the reactor core by means of a system of non linear equations, which is solved iteratively. To obtain a set of dominant modes it is necessary to approximate the initial differential eigenvalue problem (1), by a generalized algebraic eigenvalue problem. This can be done, for example, using a finite element method [7] or a finite difference method [8]. On the other hand, for nuclear reactors the spatial mesh is naturally defined by the different materials defining the core, and for this reason, it is interesting to use a method that uses a fixed mesh and increases its accuracy without changing this mesh.

Spectral Methods are based on approximating the solution of the problem as a truncated expansion in terms of a suitable basis of polynomials. The accuracy obtained in the solution is controlled by means of the number of polynomials considered on the expansion and it is not necessary to refine the mesh to increase the accuracy of the method. Here we present a high order finite element method that uses as a basis of polynomials the modified Dubiner's polynomials [9, 10], and increases its accuracy increasing the number of polynomials used in the solution expansions, leaving the initial mesh unaltered.

## 2. DISCRETIZATION OF THE PROBLEM

The first step for the discretization of a bidimensional reactor with hexagonal geometry, is to divide each hexagon into six equilateral triangles. We will denote by $\Omega_e$ each one of these triangles of the discretization. To show the development of the method, we consider the Lambda Modes problem in the one-group energy approximation

$$-\vec{\nabla} D(x,y) \cdot \vec{\nabla}\Phi(x,y) + \Sigma_a(x,y)\Phi(x,y) = \frac{1}{\lambda}\nu\Sigma_f(x,y)\Phi(x,y) \ , \tag{2}$$

where standard notation has been used [1]. The generalization of this process to more energy groups is straightforward.

### 2.1. Variational formulation

Without loss of generality, the neutron diffusion equation (2) can be written as

$$-\vec{\nabla} D(x,y) \cdot \vec{\nabla}\Phi(x,y) + \Sigma_a(x,y)\Phi(x,y) = S(x,y) \ , \tag{3}$$

where $\Phi(x,y)$ is the neutron flux, and the source term, $S(x,y)$, is defined as follows,

$$S(x,y) = \frac{1}{\lambda}\nu\Sigma_f(x,y)\Phi(x,y) \ . \tag{4}$$

If the region defined by the reactor core is called $\Omega$, to solve the problem defined by equation (3) boundary conditions over the boundary of the reactor, $\partial\Omega$, are needed. The polygonal defining $\partial\Omega$ is split into two components, $\partial\Omega_0$ and $\partial\Omega_\beta$. $\partial\Omega_0$ is the component of this polygonal with zero flux boundary condition and $\partial\Omega_\beta$ is the component with albedo boundary condition. Formally, this albedo boundary condition can be written as [11]

$$D(x,y)\vec{n}(x,y) \cdot \vec{\nabla}\Phi(x,y) + \frac{1}{2}\frac{1-\beta}{1+\beta}\Phi(x,y) = 0 \ , \ \ (x,y) \in \partial\Omega_\beta \ , \tag{5}$$

where $\vec{n}(x,y)$ is the normal vector to the boundary in the outward direction.

There are different variational formulations for which the neutron diffusion equation is a stationary point of a suitable functional in a Sobolev space. The functional that will be used in the following is [12]

$$\begin{aligned}\mathcal{F}(\Phi) &= \frac{1}{2}\iint_{\Omega} D(x,y)\vec{\nabla}\Phi(x,y)\cdot\vec{\nabla}\Phi(x,y)dxdy + \frac{1}{2}\iint_{\Omega}\Sigma_a(x,y)\Phi^2(x,y)dxdy \\ &- \iint_{\Omega} S(x,y)\Phi(x,y)dxdy + \int_{\partial\Omega_\beta}\frac{1}{4}\frac{1-\beta}{1+\beta}\Phi^2(x,y)dl\ . \end{aligned} \tag{6}$$

Thus, $\Phi(x,y)$ is a stationary point of functional (6) with respect to an arbitrary variation $\delta\Phi(x,y)$ if, and only if, the following Euler conditions are fulfilled [12]:

1. The neutron flux satisfies the condition

$$-\vec{\nabla}D(x,y)\cdot\vec{\nabla}\Phi(x,y) + \Sigma_a(x,y)\Phi(x,y) = S(x,y)\ , \tag{7}$$

   which is identical to the neutron diffusion equation (3).

2. The neutron flux satisfies the equation

$$D(x,y)\vec{n}(x,y)\cdot\vec{\nabla}\Phi(x,y) + \frac{1}{2}\frac{1-\beta}{1+\beta}\Phi(x,y) = 0\ , \tag{8}$$

   if $(x,y)\in\partial\Omega_\beta$, which is identical to the albedo type boundary condition given by equation (5).

3. The neutron current must be continuous over the internal faces of the subdomains in the perpendicular direction. This can be written as

$$D_e\vec{\nabla}\Phi(x,y)\cdot\vec{n}_e(x,y) = -D_{e'}\vec{\nabla}\Phi(x,y)\cdot\vec{n}_{e'}(x,y). \tag{9}$$

### 2.2. Reference element

To develop the method, we use a change of variable transforming each one of the triangular elements of the mesh, $\Omega_e$, of the mesh into the *Reference domain*, $\Omega_{REF}$, also known in the literature as the *Right Triangle* (see Figure 1).

This change of variables, relating physical coordinates, $(x,y)$, with the coordinates of the Reference domain, $(x',y')$, is given by

$$\left\{\begin{array}{rl} x = & x_0 + (x_1-x_0)x' + (x_2-x_0)y' \\ y = & y_0 + (y_1-y_0)x' + (y_2-y_0)y' \end{array}\right. \quad\Leftrightarrow\quad \left\{\begin{array}{rl} x' = & a_1x + b_1y + c_1 \\ y' = & a_2x + b_2y + c_2 \end{array}\right. \tag{10}$$

where $a_i, b_i, c_i$ are constants depending on the coordinates of the vertices of each triangle.

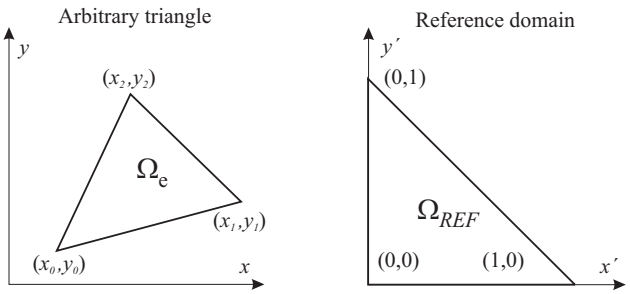



**Figure 1. Change of variables from a arbitrary triangle to the reference domain.**

We will denote by $\Phi_e$ the restriction of the neutron flux, $\Phi$, to the element $\Omega_e$ of the mesh, and we will assume that the solution of problem (2) can be approximated by a finite expansion of the form,

$$\Phi_e(x', y') = \sum_{i,j=0}^{i+j\leq K} \phi_{e,ij} g_{ij}(x', y') \ , \tag{11}$$

where $g_{ij}$ are elements of a polynomial basis in terms of the coordinates $(x', y')$ of the reference domain.

The aim of the discretization method is to obtain a generalized algebraic eigenvalue problem that approximates the differential eigenvalue problem (2), being the eigenvectors, vectors whose components are related with the coefficients $\phi_{e,ij}$ for all the triangles of the mesh.

### 2.3. Polynomial basis

Any physical solution for the neutronic flux has to be a continuous function on the region defining the reactor core, $\Omega$. To obtain this kind of solutions, we will use as a basis of polynomials the modified Dubiner's polynomials [9, 10], and we will select some of the coefficients of the expansions (11) in such a way that any function expanded in terms of these polynomials is continuous by construction. Modified Dubiner's polynomials are defined on the Reference domain (see Fig. 1), and we distinguish among vertex polynomials, edge polynomials, and interior polynomials (see in Figure 2). The definitions of these polynomials are the following ones [10]:

- Vertex polynomials

$$\begin{aligned} g_{11}(x', y') &= 1 - x' - y' \ , && \text{(vertex 1)}, \\ g_{21}(x', y') &= x' \ , && \text{(vertex 2)}, \\ g_{31}(x', y') &= y' \ , && \text{(vertex 3)}, \end{aligned}$$

- Edge polynomials $(2 \leq j \leq K)$

$$g_{1j}(x', y') = (1 - x' - y')\, x'\, Q^{1,1}_{j-2}(x', y')\; 2^{\frac{3}{2}}\,, \qquad \text{(edge 1)},$$
$$g_{2j}(x', y') = x'\, y'\, P^{1,1}_{j-2}(2y' - 1)\; 2^{\frac{3}{2}}\,, \qquad \text{(edge 2)},$$
$$g_{3j}(x', y') = y'\; (1 - x' - y')\; P^{1,1}_{j-2}(1 - 2y')\; 2^{\frac{3}{2}}\,, \qquad \text{(edge 3)},$$

- Interior polynomials $(4 \leq i,\ 1 \leq j,\ i + j \leq K + 2)$

$$g_{ij}(x', y') = (1 - x' - y')\, x'\, y' Q^{1,1}_{i-4}(x', y')\, P^{2i-5,1}_{j-1}(2y' - 1)\, 2^j\,, \quad \text{(interior)}.$$

where

$$Q^{1,1}_i(x', y') = P^{1,1}_i\left(\frac{2x'}{1 - y'} - 1\right)(1 - y')^i\,,$$

and $P^{\alpha,\beta}_i(z)$ are the Jacobi polynomials, which are orthogonal polynomials in $[-1, 1]$.

To simplify the notation, polynomials $g_{mn}(x', y')$ will be denoted with only one subindex $i$, $g_i(x', y')$, with $i = 1, \ldots, M_K = (K + 1)(K + 2)/2$. The order defined by the index $i$ is obtained numbering for a given element, $e$, first the polynomial of vertex 1, after this, the polynomials of the edge 1 in ascent order with the polynomial degree, after the second vertex polynomial followed of the polynomials of the second edge with the same criterion, the third vertex polynomial and the polynomials of the third edge with the same order as before, and then the interior polynomials (see in Figure 2).

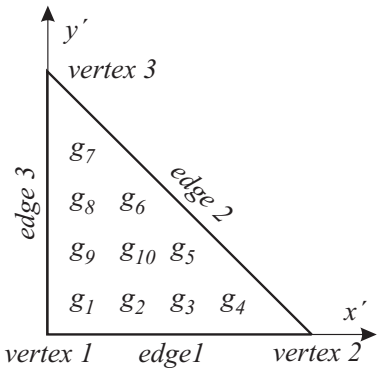



**Figure 2. Numbering the polynomials.**

We define these polynomials over each element of the spatial discretization, $\Omega_e$, by means of the change of variables (10), and we rename the coefficients being coherent with the notation used for the polynomials. In this way, $g_{e,i}(x', y')$, the polynomial $i$ over the element $\Omega_e$, is associated to the coefficient $\phi_{e,i}$. These coefficients will be the *local* unknowns of the problem.

Following this notation, we write the neutron flux over each element defining the reactor domain in the physical coordinates $(x, y)$ and the local coordinates $(x', y')$ as

$$\Phi_e(x,y) = \sum_{i=1}^{M_K} \phi_{e,i} g_{e,i}(x,y) \ , \quad \Phi_e(x',y') = \sum_{i=1}^{M_K} \phi_{e,i} g_i(x',y') \ . \tag{12}$$

Now, we establish relations among coefficients of some polynomials corresponding to adjacent triangles, to assure the continuity of the neutron flux. Eliminating the redundant coefficients of expansions (11), we obtain a reduction in the number of unknowns necessaries to describe the problem. This new set of coefficients will be called the *global unknowns*. To explain this process, we use an example shown in Figure 3. In this Figure, a typical configuration for two adjacent interior triangles, $\Omega_{e_1}$ and $\Omega_{e_2}$, using a degree $K = 2$ in the polynomial expansions is shown.

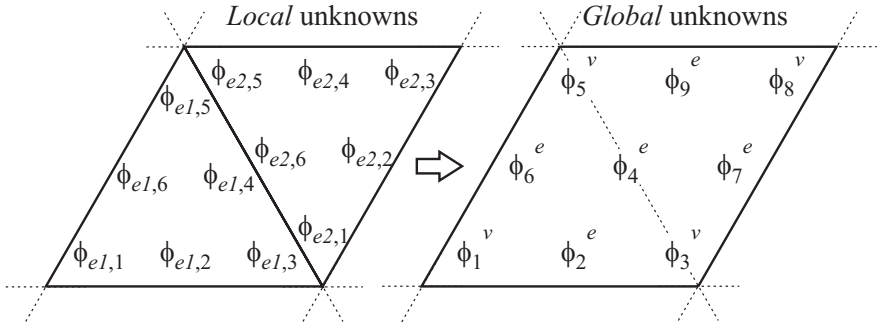



**Figure 3. Continuity conditions for vertex and edge polynomials for $K = 2$.**

Given a vertex of a triangle, only the corresponding vertex polynomial takes a value different from zero on this vertex. Thus, the continuity of the neutron flux at this vertex can be assured fixing the coefficient of the corresponding vertex polynomial on a first triangle, and making the coefficients associated to the same vertex polynomial of the others triangles, that have this vertex in common, equal to the coefficient of the first triangle. For this reason, for each internal vertex of the mesh, we have only one *global* unknown to be determined. In the same way, for a given edge of a triangle, the only polynomials that take a value different from zero on this edge are the corresponding edge polynomials. The continuity for the neutron flux at the inner edges of the mesh is assured fixing the coefficients of the edge polynomials of the first triangle and making the corresponding coefficients of the edge polynomials of the adjacent triangle equal to the fixed edge coefficients for the first triangle.

The reduction of unknowns from the local unknowns is carried out with the aid of an order array relating the local unknowns with the global unknowns after the reduction process. An example

for $K = 2$ with two adjacent triangles is shown in equation (13),

$$\begin{array}{c} \textit{Local} \text{ unknowns} \\ \left[\begin{array}{c} \phi_{e_1,1} \\ \phi_{e_1,2} \\ \phi_{e_1,3} \\ \phi_{e_1,4} \\ \phi_{e_1,5} \\ \phi_{e_1,6} \\ \hline \phi_{e_2,1} \\ \phi_{e_2,2} \\ \phi_{e_2,3} \\ \phi_{e_2,4} \\ \phi_{e_2,5} \\ \phi_{e_2,6} \end{array}\right] \left[\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \hline 3 \\ 7 \\ 8 \\ 9 \\ 5 \\ 4 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \textit{Global} \text{ unknowns} \\ \left[\begin{array}{c} \phi_1^v \\ \phi_2^e \\ \phi_3^v \\ \phi_4^e \\ \phi_5^v \\ \phi_6^e \\ \phi_7^e \\ \phi_8^v \\ \phi_9^e \end{array}\right] \left[\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}\right] \end{array} \tag{13}$$

where $\phi_j^v$ refer to the global vertex coefficient associated with an interior vertex and $\phi_j^e$ refer to global edge coefficients corresponding to an interior edge. The coefficients corresponding to interior polynomials are unaltered after the reduction process.

In the following, we will denote by $\phi_j^i$ the global interior coefficients corresponding to an interior polynomial, and by $\phi_j^b$ the global boundary coefficient associated with a boundary polynomial, i.e., the polynomials which are different to zero at the boundary $\Omega_\beta$ of the reactor core.

### 2.4. Building the matrices

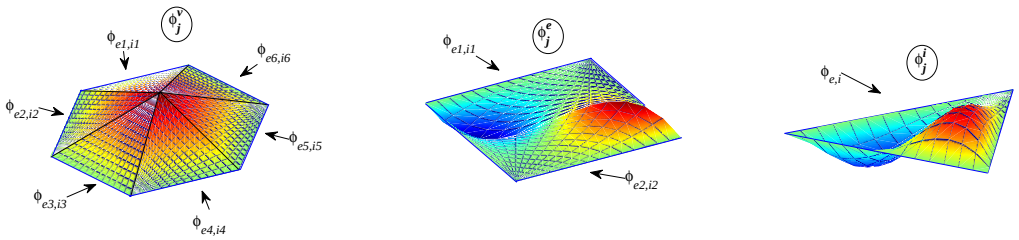
The equations defining the algebraic eigenvalue problem that approximates the Lambda Modes Problem are obtained using expansions (12) for the neutron flux in the functional (6), and making equal to zero the derivatives respect to the global coefficients. We must distinguish among derivatives with respect to the coefficients associated to an interior vertex, to an interior edge, or to a vertex and an edge belonging to $\partial\Omega_\beta$.

When we consider a coefficient associated with an internal vertex, $\phi_j^v$, the only polynomials that will be different to zero at this vertex are the vertex polynomials corresponding to this vertex (see Fig. 4(a)), for each triangle containing this vertex. Thus, when we derive respect to $\phi_j^v$, which is equal to the local coefficients $\phi_{e_1,i_1}$, $\phi_{e_2,i_2}$, $\phi_{e_3,i_3}$, $\phi_{e_4,i_4}$, $\phi_{e_5,i_5}$ y $\phi_{e_6,i_6}$ we obtain an equation of the form

$$\sum_{(e,i)\in\mathcal{I}_j^v} \left( D_e \iint_{\Omega_e} \vec{\nabla} g_{e,i} \cdot \vec{\nabla}\Phi_e \, dx \, dy + \Sigma_{a,e} \iint_{\Omega_e} g_{e,i}\Phi_e \, dx \, dy - \frac{1}{\lambda}\nu\Sigma_{f,e} \iint_{\Omega_e} g_{e,i}\Phi_e \, dx \, dy \right) = 0 \,, \tag{14}$$

where $\mathcal{I}_j^v = \{(e_1, i_1), (e_2, i_2), (e_3, i_3), (e_4, i_4), (e_5, i_5), (e_6, i_6)\}$, the set of pairs of indices that contain the six vertex polynomials showed in Figure 4(a).

When we derive with respect to an internal edge coefficient, $\phi_j^e$, shared by the elements $\Omega_{e_1}$ and $\Omega_{e_2}$, there are two edge polynomials that have their coefficients fixed to this global edge




(a) Interior vertex (b) Interior edge (c) Interior element

**Figure 4. Global unknowns associated with an interior vertex (a), an interior edge (b), and an interior polynomial (c), where the global coefficient is that inside the circle.**

coefficient (see Fig. 4(b)), and we obtain

$$\sum_{(e,i)\in\mathcal{I}^e_j}\left(D_e\iint_{\Omega_e}\vec{\nabla}g_{e,i}\cdot\vec{\nabla}\Phi_e\,dx\,dy+\Sigma_{a,e}\iint_{\Omega_e}g_{e,i}\Phi_e\,dx\,dy-\frac{1}{\lambda}\nu\Sigma_{f,e}\iint_{\Omega_e}g_{e,i}\Phi_e\,dx\,dy\right)=0\,, \tag{15}$$

where $\mathcal{I}^e_j=\{(e_1,i_1),(e_2,i_2)\}$ is the set of pairs of indices that contain the two edge polynomials showed in Figure 4(b).

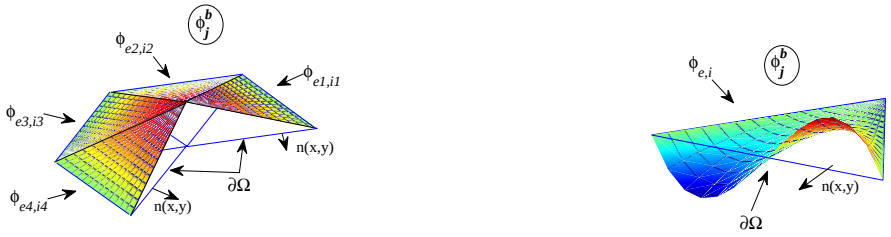
The coefficient associated to an interior polynomial is itself a global interior coefficient (see Figure 4(c)) and, thus, when we take the derivatives with respect to a global interior coefficient we obtain

$$D_e\iint_{\Omega_e}\vec{\nabla}g_{e,i}\cdot\vec{\nabla}\Phi_e\,dx\,dy+\Sigma_{a,e}\iint_{\Omega_e}g_{e,i}\Phi_e\,dx\,dy=\frac{1}{\lambda}\nu\Sigma_{f,e}\iint_{\Omega_e}g_{e,i}\Phi_e\,dx\,dy\,. \tag{16}$$

If we deal with zero flux boundary conditions, the global boundary coefficients associated with polynomials that do not take the value zero over $\Omega_0$ are set to zero. On the other hand, if we deal with albedo type boundary conditions, we take the derivatives with respect to the boundary global coefficients, associated with their respective vertex or edge polynomials (see Figure 5), obtaining equations of the form

$$\begin{aligned}&\sum_{(e,i)\in\mathcal{I}^b_j}\left(D_e\iint_{\Omega_e}\vec{\nabla}g_{e,i}\cdot\vec{\nabla}\Phi_e\,dx\,dy+\Sigma_{a,e}\iint_{\Omega_e}g_{e,i}\Phi_e\,dx\,dy=\right.\\&\left.\frac{1}{\lambda}\nu\Sigma_{f,e}\iint_{\Omega_e}g_{e,i}\Phi_e\,dx\,dy+\int_{\partial\Omega_\beta}\frac{1}{2}\frac{1-\beta}{1+\beta}g_{e,i}\Phi_e\,dl\right)=0\,,\end{aligned} \tag{17}$$

where $\mathcal{I}^b_j=\{(e_1,i_1),(e_2,i_2),(e_3,i_3),(e_4,i_4)\}$ in the case of a global boundary coefficient such as the one shown on Figure 5(a), or $\mathcal{I}^b_j=\{(e,i)\}$ in the case of a global boundary coefficient such as the one showed on the Figure 5(b).




(a) Boundary vertex (b) Boundary edge

**Figure 5. Global unknowns associated with a boundary vertex (a) and with a boundary edge (b), where the global coefficient is that inside the circle.**

With the discretization method exposed above, we approximate the two energy groups approximation of the neutron diffusion equation (1) by a generalized algebraic eigenvalue problem with the following block structure

$$\begin{bmatrix} L_{11} & 0 \\ -L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} M_{11} & M_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

where $\phi_1$ is a vector with the global unknowns corresponding to the neutron flux for the fast group, and $\phi_2$ is a vector with the global unknowns of the thermal flux.

To solve this problem, it is reduced to the ordinary eigenvalue problem

$$L_{11}^{-1}\left(M_{11} + M_{12}L_{22}^{-1}L_{21}\right)\phi_1 = \lambda\phi_1 \ ,$$

which is solved for the dominant eigenvalues and their corresponding eigenvectors using the Implicit Restarted Arnoldi method [13].

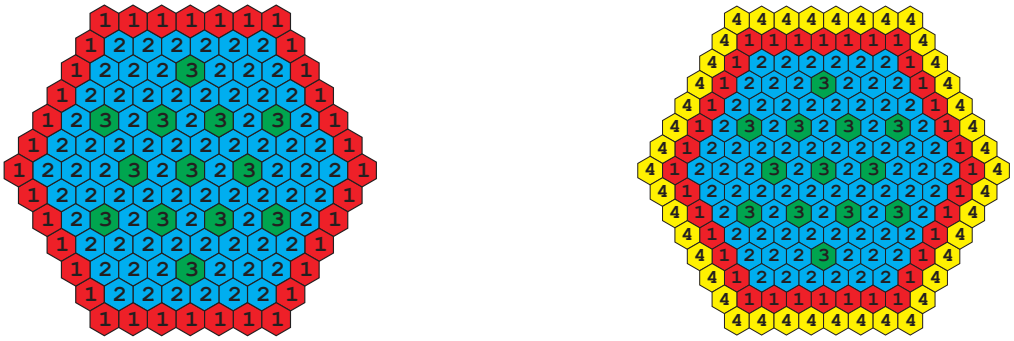
## 3. NUMERICAL RESULTS

To test the performance of the spectral element method, we have computed the dominant modes of different 2D benchmark reactor problems presented in [5]. In particular, we have studied the IAEA with and without reflector problems. We take as a reference the results of the $k_{\text{eff}}$ and the power distribution computed using the finite difference code DIF3D, where a mesh of 864 triangles per hexagon was used. For all the problems we will show tables with the different values of the $k_{\text{eff}}$ obtained with different values of the degree $K$ used in the neutron flux expansions. Also the difference between the obtained result and the reference value are presented. This difference is denoted as $\Delta k_{\text{eff}}$ in $pcm$ (percent-milli, i.e., $10^{-5}$).

In order to compare the performance of the method with other existing finite element methods, we have also computed the $k_{\text{eff}}$ and the neutronic power distribution using the TRIVAC code [7]. TRIVAC calculations have been performed for two representative configurations, $T_1$ and $T_2$. $T_1$

configuration has been obtained setting the polynomial order equal to $1$, $3$ lozenges per hexagon and the Gauss-Legendre quadrature rule for the integration. $T_2$ has been obtained setting the polynomial order equal to $3$, with $12$ lozenges per hexagon and the Gauss-Legendre quadrature rule. Both the Spectral Element Method (SEM) and the TRIVAC code have been run in a personal computer with 2 processors Intel Core 2, of 1.86 GHz and with 1GB of RAM. In the $k_{\text{eff}}$ tables, results of CPU times needed for the computations have been reported. We also present results of the subcritical eigenvalues and the symmetry patterns of the power distribution associated with the subcritical modes computed with the spectral element method, for each one of the reactor problems.

### 3.1. IAEA with and without reflector problems



(a) IAEA without reflector. (b) IAEA with reflector.

**Figure 6. Geometry of the IAEA without reflector (a) and with reflector (b) problems.**

This problem has 13 control rods inserted, and has a $1/12$ reflective symmetry, as it is shown in Figure 6(a) but, as the subcritical modes do not maintain this symmetry, the computations are performed for all the problems considering the whole reactor. The assembly pitch is of $20.0$ cm. The reflector is not included in the core, and we will consider vacuum boundary conditions, that is, albedo boundary conditions with $\beta = 0$. The nuclear cross sections for this problem are presented in [5].

The IAEA with reflector problem has been also studied. This problem is the same as the previous one, except that in this problem an additional layer of reflector surrounding the core is included, as shown in Figure 6(b). Also, vacuum boundary conditions are considered.

Table I shows the different results obtained for the $k_{\text{eff}}$ for both IAEA with and without reflector problems for the SEM method, for the TRIVAC code and the reference results obtained from [5].

Figure 7 shows the neutronic power distribution for each hexagon computed with the SEM method using a degree $K = 6$ in the expansions for the IAEA problem without reflector, together with the reference result and the percentage of the relative error on each hexagon.

Also, Figure 8 shows the neutronic power distribution for each hexagon computed with the SEM method using a degree $K = 6$ in the flux expansions for the IAEA problem with reflector,

**Table I. $k_{\text{eff}}$ results for the IAEA reactor without reflector (left) and with reflector (right).**

| | IAEA Without Reflector | | | IAEA With Reflector | | |
|---|---|---|---|---|---|---|
| $K$ | $\lambda_1(k_{\text{eff}})$ | $\Delta k_{\text{eff}}(pcm)$ | CPU time (s) | $\lambda_1(k_{\text{eff}})$ | $\Delta k_{\text{eff}}(pcm)$ | CPU time (s) |
| 3 | 0.9780083 | 6.8649 | 0.671 | 1.0055754 | −6.8360 | 1.015 |
| 4 | 0.9780645 | 1.2446 | 1.265 | 1.0055166 | −0.9680 | 2.015 |
| 5 | 0.9780735 | 0.3492 | 2.655 | 1.0055102 | −0.3270 | 4.108 |
| 6 | 0.9780754 | 0.1604 | 4.280 | 1.0055096 | −0.2590 | 6.655 |
| $T_1$ | 0.9766445 | 143.2500 | 0.077 | 0.9766445 | 106.5620 | 0.119 |
| $T_2$ | 0.9780765 | 0.0426 | 7.617 | 1.0055062 | 0.0711 | 10.952 |
| Ref. | 0.9780770 | – | – | 1.0055070 | – | – |

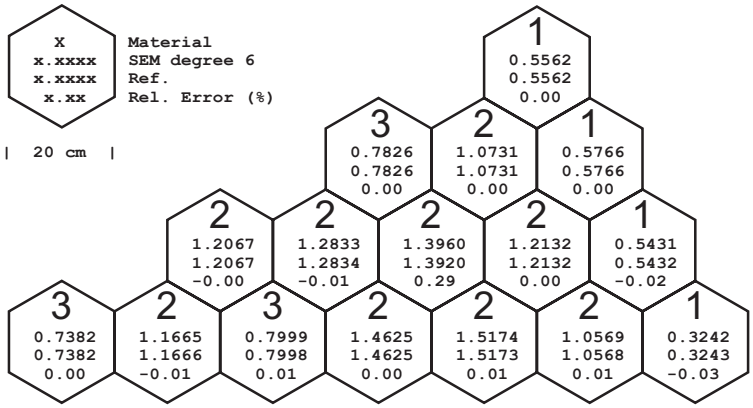



**Figure 7. Power distribution for the IAEA problem without reflector.**

together with the reference result and the percentage of the relative error on each hexagon.

In Table II, the results obtained for the first three subcritical eigenvalues computed with the SEM method using different values of the degree $K$ are presented, for both IAEA without reflector (left) and for the IAEA with reflector (right) problems.

A symmetry pattern for the power distribution associated with the four dominant modes of each one of these problems are shown in Figure 3.1, where the symmetry patterns for each one of the dominant modes is the same for the two problems .

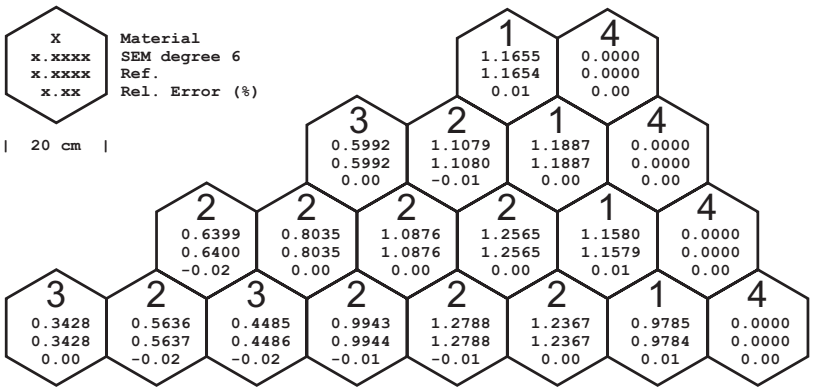
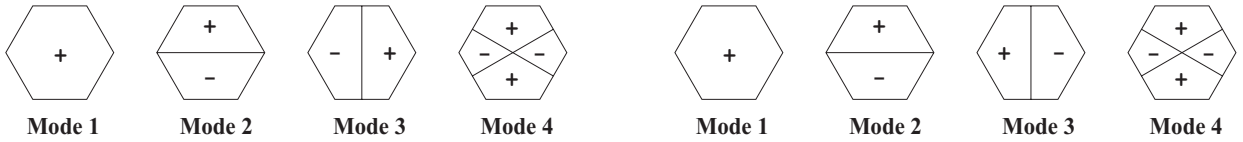



**Figure 8. Power distribution for the IAEA problem with reflector.**

**Table II. First 3 subcritical eigenvalues for the IAEA problem with reflector (left) and without reflector (right).**

| | IAEA Without Reflector | | | IAEA With Reflector | | |
|---|---|---|---|---|---|---|
| $K$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
| 3 | 0.9631196 | 0.9631196 | 0.9383851 | 0.9965746 | 0.9965746 | 0.9768964 |
| 4 | 0.9631683 | 0.9384269 | 0.9196477 | 0.9964991 | 0.9964991 | 0.9768022 |
| 5 | 0.9631771 | 0.9631771 | 0.9384355 | 0.9964908 | 0.9964908 | 0.9767917 |
| 6 | 0.9631791 | 0.9631791 | 0.9384376 | 0.9964899 | 0.9964899 | 0.9767904 |




(a) IAEA problem without reflector

(b) IAEA problem with reflector

**Figure 9. Symmetries for the four dominant modes for each one of the problems.**

## 4. CONCLUSIONS

A high order finite element method for the Lambda modes problem of a reactor core in hexagonal geometry has been developed. This kind of method is also known as a spectral element method (SEM). This method is based on dividing each of the hexagons associated with the fuel bundle assemblies into six equilateral triangles, obtaining a triangular mesh that is kept fixed. The SEM

method is based on the expansion of the neutronic flux in terms of the modified Dubiner's polynomials and the accuracy of the method is improved increasing the degree of the polynomials considered in the expansions.

To test the performance of the method, an extensive analysis of 2D benchmark reactor cores has been carried out and the obtained results have been compared with the computations performed with the TRIVAC code, which is a code based on an alternative finite element method for cores with hexagonal geometry. In all the studied reactors the SEM method has provided good results for the $k_{\text{eff}}$ and the neutron power distribution and also has succeeded computing the subcritical modes of each reactor.

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