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GENERALIZED FINITE ELEMENTS

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ABSTRACT

Triangles and rectangles are the ubiquitous elements in finite element studies. Only these elements admit polynomial basis functions. Rational functions provide a basis for elements having any number of straight and curved sides. Numerical complexities initially associated with rational bases precluded extensive use. Recent analysis has reduced these difficulties and programs have been written to illustrate effectiveness. Although incorporation in major finite element software requires considerable effort, there are advantages in some applications which warrant implementation. An outline of the basic theory and of recent innovations is presented here.

Key Words: finite elements, rational basis functions, algebraic elements

1. OBJECTIVE

There are many occasions when one wishes to model a region with a covering of non-overlapping elements. Within each element one may now seek a basis for polynomials of any prescribed degree while maintaining continuity across element boundaries so as to achieve global C^0 approximation. The finite element method, for example, uses a variational principle to compute a patchwork polynomial approximation to a continuous function that solves a physical problem. The two properties of local polynomial approximation and global continuity crucial for such computation restrict use of polynomial bases to simple element geometry. For example, consider a line segment (A, B) between vertices A and B of a planar element with degree-one basis functions associated with vertices A and B and no other bases on line (A, B). For the approximation to be uniquely defined on this line by values at A and B, bases W_A and W_B must be linear on (A, B) and zero along all edges not containing vertices A and B. It will be shown that this limits polynomial bases to triangle and parallelogram elements in general. We seek bases that admit more general geometries and discover that rational functions enable application to a much wider class of elements.

Two numerical considerations are of great concern. The first is complexity of basis function construction. The second is evaluation of integrals of products of basis functions and their derivatives for finite element discretization. Polynomial bases over simple element geometries provide a benchmark for computational complexity. Early application of rational bases with more complex geometries was limited by these considerations. One may choose a large enough number of triangle and parallelogram (usually rectangular) elements to achieve a desired accuracy. Disadvantages of fine partitioning include the large amount of data to be stored and processed and the increased difficulty in solution of the discretized equations. Improved efficiency and memory of modern scientific computers has extended application to problems with hundreds of thousdands of elements. A major goal is to reduce storage and computation time by admitting more general element geometry.

2. EARLY DEVELOPMENT OF RATIONAL BASIS FUNCTIONS

Finite element computation has widespread application. Elements with polynomial basis functions are restricted to three or four sides (straight or isoparametric parabolas) in two space dimensions and four sided tetrahedra or six sided rectangular parallelepipeds in three space dimensions. That polynomials do not suffice for more general elements is easily shown. Values at vertices are free parameters for finite element variational discretization. For the simplest patchwork degree-one and global C^0 analysis the element approximation is linear along each side connecting adjacent vertices. First, we consider triangular elements. Straight sided triangular elements are acceptable. Otherwise, if a triangle has one or more curved sides, polynomial bases are adequate for only a few exceptional geometries. In general, sides will intersect at one or more points other than the vertices. The extensions of the single valued linear approximation within the element along the sides is in general double valued at all simple exterior intersection points. Polynomials are single valued and thus cannot provide a basis for approximation in such cases. This is also the case for any element with exterior intersection points (hereafter abbreviated as EIP). Thus, straight-sided triangles and parallelograms are the only elements which admit polynomial basis functions. The simplest generalization of polynomials is to rational bases with denominator polynomials that vanish at all EIP.

A simple algebraic curve is a curve on which an irreducible polynomial vanishes. The curve is rational when it has a rational parametrization, in which case it is said to be of genus zero. An element is rational when all of its bounding sides are rational. All lines and conics are rational. This discussion will be restricted to rational algebraic elements. The term rational refers to parametrized curves and to a ratio of polynomials. Hereafter it will be used only in the latter sense.

Thirty-five years ago I developed theory [1] for construction of rational basis functions for well-set algebraic elements with an arbitrary number of straight or curved sides in two or three space dimensions. An element is well-set when the algebraic curve of each side has only simple points on the side and no side (face) extends to intersect the element at any point not on the element side (face). We consider first two-dimensional elements. An element with n sides is called an n-pol. If the sum of the orders of its sides is m it is an n-pol of order m. When all sides are straight $n = m$ and the n-pol is a convex polygon. When curved sides are restricted to conic it is an n-con. The rational basis functions for any degree approximation have a common denominator, Q, of degree not greater than $m - 3$ uniquely determined by the EIP, and in [1] Q was actually calculated from these points. This is not a simple task when m is large, and draws heavily on algebraic geometry theory. This denominator is known as the adjoint of the element.

There are element nodes at all vertices and at another point on each conic side. Vertex values at the ends of a linear side determine a unique linear variation along the side by virtue of the linear relationship between x and y along the line. On a conic side the value at an additional point is needed to fix the linear variation needed for continuity of the approximation across the side. Similarly, for degree-two approximation a side node is required on each linear side. There are six degrees of freedom in a quadratic. However, there is one quadratic relationship on a conic side which leaves only five degrees of freedom for a general quadratic along a conic side. Thus, three side nodes suffice in addition to the two vertex nodes.

For degree one approximation, the numerator P_i is easily determined. To simplify this discussion, we restrict elements to n-cons (only linear and conic sides). At vertex i, P_i is a product of an opposite factor F_i and an adjacent factor R_i. The opposite factor is just the product of the linear and quadratic forms (arbitrarily normalized) that vanish on all sides other than the two sides which meet at i. When these two sides are lines, F_i is of degree $m-2$ and $R_i = 1$. When one side is linear and the other is conic F_i is of degree $m-3$. Node p is on the conic side and the sides meet at EIP r. The adjacent factor R_i is the arbitrarily normalized linear form $(p;r)$ that vanishes on line (p,r). When both sides are conic F_i is of degree $m-4$. There are nodes p and q on the conic sides and EIP r, s and t where the two conics meet other than at the vertex. The adjacent factor R_i is now the unique (except for arbitrary normalization) quadratic factor $(p;q;r;s;t)$ that vanishes on the conic (p,q,r,s,t). (When r, s and t are not distinct the theory of intersection sets provides three conditions which yield a unique conic [1].) At side node i, $P_i = F_i$ is the product of polynomials that vanish on all sides other than that on which i lies. In all cases P_i is of degree $m-2$. P_i may be normalized to unity at i.

In three dimensions F_i is the product of the polynomials that vanish on the faces not meeting at vertex i. In [1] elements were restricted to have only three faces meeting at each vertex. This restriction has recently been removed. For degree one approximation over restricted elements, the adjacent factor is unity at vertex i where three planes meet. When two planes and a quadric surface meet R_i is the linear form which vanishes on plane (p,q,r) where points p and q are on the conic edges meeting at i, and r is the exterior triple point (ETP) at which the linear edge pierces the quadric. The analysis developed in [1] becomes more complex when two or three quadrics meet at a vertex and will not be discussed here.

Higher degree approximation over algebraic elements is also described in detail in [1]. Node placement to assure global C^0 and any degree patchwork approximation is always consistent with nodes required for unique generation of basis functions,

3. THE *GADJ* ALGORITHM FOR GENERATING THE ADJOINT, Q

For many years, Q was constructed directly from the EIP of an element. This was not a simple task, especially when the EIP were not distinct or when curves were nearly parallel. Gautam Dasgupta [2] developed an alternative method for convex polygons which greatly simplified the determination of Q. I subsequently generalized this algorithm to n-cons and to three space dimensions [3]. This GADJ algorithm will now be described.

Basis function numerators P_i of degree $m-2$ are constructed easily. Curved sides require construction of adjacent factors from EIP of adjacent sides only as described in Section 2. Degree one approximation requires that the sum of the basis functions be unity. It follows that for normalization coefficients k_i to be determined for basis functions $W_i = k_i \frac{P_i}{Q}$:

$$\sum_{i=1}^{m} k_i P_i = Q. \tag{1}$$

The GADJ algorithm sets $k_1 = 1$ and determines the remaining k_i recursively. Dasgupta's analysis [2] for convex polygons will now be described. Consider the side $(j, j+1)$ between vertices j and $j+1$. The numerator P_i is zero on this side for all i other than j and $j+1$. Also, W_j is linear on side $(j, j+1)$. Therefore,

$$\frac{k_j P_j}{k_j P_j + k_{j+1} P_{j+1}} \tag{2}$$

must be linear on side $(j, j+1)$. Factors common to P_j and P_{j+1} may be divided out. Let $(s; s+1)$ denote the linear form that vanishes on side $(s, s+1)$, and let its value at i be denoted as $(s; s+1)_i$. Then

$$\frac{k_j(j+1; j+2)}{k_j(j+1; j+2) + k_{j+1}(j-1; j)} \tag{3}$$

is linear on $(j, j+1)$. (At node $j = 1$, $j - 1 = n$.) The numerator is linear. Hence, k_{j+1} must be chosen so that the denominator is constant on $(j, j+1)$:

$$k_{j+1} = k_j \frac{(j+1; j+2)_j}{(j-1; j)_{j+1}}. \tag{4}$$

The generalization to n-cons may be found in [3]. Algebraic geometry theorems relating to congruences of products of linear forms to quadratic forms on lines and conics play an essential role in the development. For example, consider quadratic $P_2(x, y)$ on conic side s. The quadratic vanishes on conic c which intersects s at points (A, B, C, D). The product of linear forms $(A; B)$ and $(C; D)$ also intersect s at these four points. A fundamental theorem of algebraic geometry asserts that two polynomials that vanish at the same points on a curve are congruent on the curve. Thus, if P_2 and $(A; B)(C; D)$ are normalized to be equal at any point on s they are equal at all points on s. Use of such congruences is illustrated by considering the numerators of basis functions on conic side $(j, j+1)$ with side node $j + 1/2$. Let side $(j-1, j)$ with side node $j - 1/2$ be a conic that intersects $(j, j+1)$ at j and EIP A, B, C. Let side $(j+1, j+2)$ be a line that intersects $(j, j+1)$ at $j+1$ and EIP D. Then

$$N_j = k_j(j+1; j+2) R_j F \tag{5}$$

where F is the product of forms which vanish on all sides opposite vertex j other than $(j+1; j+2)$ and R_j is the adjacent factor $(j - 1/2; j + 1/2; A; B; C)$. The basis function numerator at $j+1$ is

$$N_{j+1} = k_{j+1}(j + 1/2; D) S F \tag{6}$$

where S is the quadratic that vanishes on conic side $(j-1, j)$. The intersection set of N_j/F and side $(j, j+1)$ is $\{j+1, j+1/2, A, B, C, D\}$. The product of the three lines $(j + 1/2, j+1)(A, D)(B, C)$ has the same intersection set. After normalization c_j for equality at j, we have

$$N_j = k_j c_j (j + 1/2; j+1)(A; B)(C; D) F \mod (j, j+1). \tag{7}$$

The intersection set of N_{j+1}/F with conic side $(j, j+1)$ is $\{j + 1/2, D, j, A, B, C\}$. The product of the three lines $(j, j + 1/2)(A, B)(C, D)$ has the same intersection set so that after normalization c_{j+1} at $j+1$

$$N_{j+1} = k_{j+1} c_{j+1} (j; j + 1/2)(A; B)(C; D) F \mod (j, j+1). \tag{8}$$

The GADJ algorithm then yields

$$k_{j+1} = k_j \frac{c_j(j + 1/2; j + 1)_j}{c_{j+1}(j; j + 1/2)_{j+1}}. \tag{9}$$

Similarly, with normalization $c_{j+1/2}$

$$k_{j+1/2} = k_j \frac{c_j(j + 1/2; j + 1)_j}{c_{j+1/2}(j; j + 1)_{j+1/2}}. \tag{10}$$

Correctness and robustness of GADJ has been verified with MATLAB. GADJ eliminates the more challenging task of direct generation of Q from the EIP. Construction of rational basis functions for well-set n-cons is now simple and efficient. Three-dimensional convex polyhedral elements introduce no difficulties. The recursive algorithm proceeds with planar forms $(A; B; C)$ replacing linear forms $(j; k)$. Generalization to quadric faces leads to numerical complexities and has yet to be attempted. The primary difficulty occurs when there are adjacent quadric faces. Degree-one approximation requires construction of adjacent factors from edge nodes, ETP, and singular points on quartic edges.

4. QUADRATURE

A former obstacle to implementation with rational bases was computation of integrals of weighted combinations of rational functions and their derivatives for generation of the finite element difference equations. Dasgupta resolved this by using the divergence theorem to convert from area to contour integrals. If $f(x, y)$ is the integrand, one first computes the common integral $g(x, y)$ with respect to y and then integrates g with respect to x around the element boundary. This contour integration of integrands of rational functions and logarithms, especially along conic sides, can become quite involved. This has been treated with numerical approximations like Gaussian and Romberg quadrature. An alternative has been studied. Polynomial contour integrands may be handled analytically. In finite element applications the integrands are derived from a variational principle to minimize a prescribed Euclidean error norm. If one chooses a different norm one can reduce rational integrands to polynomial integrands. For example, in key applications the integrands are products of basis functions and of their first partial derivatives. These lead to denominators of Q^2 and Q^4. If one weights all integrands by Q^4 the weighted integrands are all polynomials. In order to integrate constants exactly, one may compute the element area A and $K = \int(Q^4)$ and normalize all integrals by multiplying them by A/K. The adjoint is unity for triangles and rectangles so that this is the standard Euclidean norm for these elements. As a grid is refined Q becomes more nearly constant within each element. A similar situation is found in isoparametric computation where the jacobian of the transformation between local and global coordinates plays the role of Q.

5. GENERAL CONVEX POLYHEDRA

Analysis of convex polyhedra in [1] was restricted to elements all of whose vertices were of order three (three faces meeting at each vertex.) Warren [4] eliminated this restriction with a different approach. Higher order vertices were discarded in [1] because Q must vanish at such vertices.

This appeared to yield singular bases. Recent analysis stimulated by Warren's basis construction demonstrated that these were removable singularities.

Let j be a vertex of order $r > 3$ of a convex polyhedron of order m. One may replace j by plane L^j close to j with j replaced by r vertices of order three of a new convex polyhedron. If this is repeated for each vertex of order greater than three the new element has a regular basis. If we now let plane L_j approach j, the sum of the r regular basis functions on the plane approaches a regular basis function at j. This may be done for each vertex of order greater than three. The adjoint Q must be of order $r - 3$ at j. Hence, the fact that the singularity is removable implies that the numerator of the basis function at j must also be of order $r - 3$ at j. Analysis of this process led to a simple construction of the basis function at j.

The numerator at j is $P_j = F_j R_j$ where F_j is the usual opposite factor of order $m - r$ and R_j is an adjacent factor of order $r - 3$ at j. Thus, P_j is of order $m - 3$. The adjoint Q is of order $m - 4$ and is of order $r - 3$ at j. The adjacent factor at a vertex of order three is unity. A polynomial in (x, y, z) has $(r - 1)(r - 2)/2$ terms of order $r - 3$. Subtracting one for normalization yields $r(r - 3)/2$ degrees of freedom for the adjoint at j. Any plane face introduced near j which approaches j in the limiting process is bounded by a convex polygon of order r which has $r(r - 3)/2$ EIP. The adjacent factor R_j is the unique polynomial (except for normalization) with only terms of order $r - 3$ in (x, y, z) that vanishes at the EIP of any polygon of order r formed by the intersection of a plane near j with the faces adjacent to vertex j. It may be demonstrated that this factor does not depend on choice of the plane. This follows from uniqueness of the adjoint Q.

6. POTENTIAL APPLICATION

The major concern now is when one may benefit through use of the generalized elements with their added complexity. Several applications seem appropriate. These include mesh generation from engineering drawings, successive mesh refinement for multigrid solution, coarse mesh rebalancing for convergence acceleration, and coarse grids for scoping surveys. In general the extended class of elements allows more accurate representation of actual geometry with fewer elements. Curved boundaries and interior interfaces need no longer be approximated by a series of straight edges. Another application is that they may be used to generalize isoparametrics in both two and three space dimensions. These bases apply to regular elements with any number of sides or faces for which simple quadrature formulas may be developed. A library may be established for such generalized isoparametric elements. Implementation is straightforward. The versatility of isoparametrics suggests that for many applications this could preempt computation with rational bases in global coordinates.

For multigrid computation one may start with a coarse grid that has a reasonable expectation of damping low mode error components. The equations at this level may be solved directly rather than iteratively. The next level may be generated by connecting with straight lines an interior point of each element to midside nodes. This may be repeated for each grid refinement until the desired accuracy is attained. Sophisticated methods are known for local grid refinement. This approach leads to only four-sided elements for all but the initial coarse grid. Relatively few elements will be other than convex quadrilaterals for which implementation is most efficient. Alternatively, interior curved interfaces may be better modeled with curved sides on finer grids, in which case some of the four-sided interior elements will have curved sides. Efficiency of multigrid is enhanced by accurate reduction of low mode errors on the coarse level. The coarse difference equations need only be computed initially and the added complexity may be a small part of the overall computation. For each multigrid cycle the coefficient matrix at each level is fixed. The contribution from coarse element e to the right-hand side of the coarse equation at node j of e is recomputed as the sum over finer level error at each fine point p in e times the value of the coarse level basis function j at p.

Coarse mesh rebalancing may be performed as a two-level computation with direct solution on the coarse level and iterative solution on a fine level resulting from several refinements of the type used for multigrid. The coarse direct solution may be based on an additive variational principle or a multiplicative principle. The latter is conventional rebalancing while the former is standard two-level multigrid. The advantage of the additive multigrid approach is that the coarse mesh difference equations remain fixed. It is possible to combine these methods, and this requires further study. The advantage of more accurate coarse representation of low mode error components persists. Fine mesh results within each coarse element determine equations for coarse mesh multiplicative correction. These provide shape functions for coarse modeling in scoping surveys. Synthesis and modal scoping studies utilize these shapes. Again the better geometric representation on the coarse level enhances such computation.

Algebraic elements with rational basis functions pose no significant conceptual problems. Introducing such elements into major finite element programs requires considerable effort. Eventually, advantages of the generalized elements over conventional elements may justify this effort.

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