International Conference on Mathematics, Computational Methods & Reactor Physics (M&C 2009)
Saratoga Springs, New York, May 3-7, 2009, on CD-ROM, American Nuclear Society, LaGrange Park, IL (2009)

MODIFIED CHOKE FLOW CRITERION FOR THE TWO-PHASE TWO-FLUID MODEL

Suneet Singh and Vincent A. Mousseau
Idaho National Laboratory
P.O. Box 1625, Idaho Falls, ID 83415-3840
Suneet.Singh@inl.gov; Vincent.Mousseau@inl.gov

ABSTRACT

A choked condition exists when mass flow rate becomes independent of the downstream conditions. In other words, no information can propagate in the upstream direction under this condition. The real part of the solution of the characteristic equation for the model represents velocity of the signal propagation and the imaginary part is the growth (or decay) rate of that signal. Therefore, if the real part of these eigenvalues is positive then no signal propagates in the upstream direction (choosing downstream direction to be the positive direction) resulting in the choke flow. In order to develop the choke criterion, a non-dimensional form of the characteristic equation is derived for the standard two-phase two-fluid model. The equation is in the terms of a slip Mach number M_s. It can be shown that the slip Mach number is small for many applications including nuclear reactor safety simulations. The eigenvalues of the characteristic equation are obtained as a power series expansion about the point $M_s = 0$. These eigenvalues are used to develop a choking criterion for the compressible two-phase flows.

Key Words: Two-phase, Two-Fluid model, Choke Flow, Characteristic Equation

1. INTRODUCTION

The accurate prediction of the choke flow in a nuclear reactor is important in order to accurately calculate mass inventory remaining in the reactor core for removing the decay heat from the system. An approximate analytical choke flow model was developed by Trapp and Ransom (1982) [1] and used in several legacy codes [2]. The solution obtained however is only valid for very small slip velocities. Recently, choked flow criterion has been developed for the bubbly flow regime [3, 4].

One needs to solve the characteristic equation to obtain a choke flow criterion. However, the characteristic equation for the compressible two-phase two-fluid model is a quartic polynomial equation. Although, an analytical solution for the quartic equation exists, it is in the radical form and is not useful for the analysis. Therefore, a series solution for the non-dimensional characteristic equation is obtained in terms of a slip Mach number. As the slip Mach number increases one can use more terms in the series to get a more accurate solution. The results obtained are compared with the results available in the literature [1]. A choking criterion for the compressible flows is also obtained based on the abovementioned series solution for relatively higher slip Mach numbers.

2. CHARACTERISTIC EQUATION FOR THE TWO-PHASE TWO-FLUID MODEL

The governing equations for the two-phase two-fluid model are as follows [2],
Continuity Equation:

$$\frac{\partial \alpha_k \rho_k}{\partial t} + \frac{\partial \alpha_k \rho_k u_k}{\partial x} = \Gamma_k \tag{1}$$

Momentum equation (in the non-conservative form):

$$\alpha_k \rho_k \frac{\partial u_g}{\partial t} + \alpha_k \rho_k u_k \frac{\partial u_k}{\partial x} + \alpha_k \frac{\partial P}{\partial x} = S_k \tag{2}$$

Energy Equation (in the non-conservative form):

$$\alpha_k \rho_k \frac{\partial U_k}{\partial t} + P \frac{\partial \alpha_k}{\partial t} + \alpha_k \rho_k u_k \frac{\partial U_k}{\partial x} + P \alpha_k \frac{\partial u_k}{\partial x} + P u_k \frac{\partial \alpha_k}{\partial x} = E_k \tag{3}$$

In the above and subsequent equations k is either g (for the gas phase) or l (for the liquid phase). The right hand side in the above equations does not have any derivatives and hence do not effect the hyperbolicity of the equations.

Using the product rule of the differentiation, continuity equation can be written as,

$$\alpha_k \frac{\partial \rho_k}{\partial t} + \rho_k \frac{\partial \alpha_k}{\partial t} + \alpha_k u_k \frac{\partial \rho_k}{\partial x} + \rho_k u_k \frac{\partial \alpha_k}{\partial x} + \alpha_k \rho_k \frac{\partial u_k}{\partial x} = \Gamma_k \tag{4}$$

From the equation of state, one can get, $\partial P / \partial \rho_k = c_k^2$. The continuity equation then becomes,

$$\frac{\alpha_k}{c_k^2} \frac{\partial P}{\partial t} + \rho_k \frac{\partial \alpha_k}{\partial t} + \frac{\alpha_k u_k}{c_k^2} \frac{\partial P}{\partial x} + \rho_k u_k \frac{\partial \alpha_k}{\partial x} + \alpha_k \rho_k \frac{\partial u_k}{\partial x} = \Gamma_k \tag{5}$$

Note that since,

$$\alpha_g + \alpha_l = 1 \tag{6}$$

therefore,

$$\frac{\partial \alpha_l}{\partial t} = -\frac{\partial \alpha_g}{\partial t} \tag{7}$$

and a similar relationship exists for the derivatives in space.

The system of the equations consisting of Eqs. (5), (2) and (3), can be written as a generic matrix form of a quasilinear system,

$$\mathbf{A}\frac{\partial \mathbf{F}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{F}}{\partial x} = \mathbf{C} \tag{8}$$

where the matrices **A**, **B** and vector **F** for the two-phase two-fluid model are as follows,

$$\mathbf{A} = \begin{bmatrix} \alpha_g / c_g^2 & \rho_g & 0 & 0 & 0 & 0 \\ \alpha_l / c_l^2 & -\rho_l & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_g \rho_g & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_l \rho_l & 0 & 0 \\ 0 & P & 0 & 0 & \alpha_g \rho_g & 0 \\ 0 & -P & 0 & 0 & 0 & \alpha_l \rho_l \end{bmatrix} \tag{9}$$

$$\mathbf{B} = \begin{bmatrix} \alpha_g u_g / c_g^2 & \rho_g u_g & \alpha_g \rho_g & 0 & 0 & 0 \\ \alpha_l u_l / c_l^2 & -\rho_l u_l & 0 & \alpha_l \rho_l & 0 & 0 \\ \alpha_g & 0 & \alpha_g \rho_g u_g & 0 & 0 & 0 \\ \alpha_l & 0 & 0 & \alpha_l \rho_l u_l & 0 & 0 \\ 0 & P u_g & P\alpha_g & 0 & \alpha_g \rho_g u_g & 0 \\ 0 & -P u_l & 0 & P\alpha_l & 0 & \alpha_l \rho_l u_l \end{bmatrix} \tag{10}$$

and

$$\mathbf{F} = \left[p \;\; \alpha_g \;\; u_g \;\; u_l \;\; U_g \;\; U_l \right]^T \tag{11}$$

The vector, **C**, consists of the source terms in the equation and is irrelevant for the characteristics of the model.

The characteristic equation ($|\mathbf{B} - \lambda \mathbf{A}| = 0$) for the system of equations can be written as,

$$\begin{vmatrix} \alpha_g (u_g - \lambda)/c_g^2 & \rho_g (u_g - \lambda) & \alpha_g \rho_g & 0 & 0 & 0 \\ \alpha_l (u_g - \lambda)/c_l^2 & -\rho_l (u_l - \lambda) & 0 & \alpha_l \rho_l & 0 & 0 \\ \alpha_g & 0 & \alpha_g \rho_g (u_g - \lambda) & 0 & 0 & 0 \\ \alpha_l & 0 & 0 & \alpha_l \rho_l (u_l - \lambda) & 0 & 0 \\ 0 & P(u_g - \lambda) & P\alpha_g & 0 & \alpha_g \rho_g (u_g - \lambda) & 0 \\ 0 & -P(u_l - \lambda) & 0 & P\alpha_l & 0 & \alpha_l \rho_l (u_l - \lambda) \end{vmatrix} = 0 \tag{12}$$

The expansion of the above determinant yields,

$$(u_l-\lambda)(u_g-\lambda)\begin{vmatrix} \alpha_g(u_g-\lambda)/c_g^2 & \rho_g(u_g-\lambda) & \alpha_g\rho_g & 0 \\ \alpha_l(u_g-\lambda)/c_l^2 & -\rho_l(u_l-\lambda) & 0 & \alpha_l\rho_l \\ \alpha_g & 0 & \alpha_g\rho_g(u_g-\lambda) & 0 \\ \alpha_l & 0 & 0 & \alpha_l\rho_l(u_l-\lambda) \end{vmatrix}=0 \tag{13}$$

Two of the eigenvalues for the system are u_l and u_g. It is pointed out here that all eigenvalues including the above eigenvalues should be positive for the choke low.
The remaining eigenvalues of the system can be obtained by solving the following equation,

$$\begin{vmatrix} \alpha_g(u_g-\lambda)/c_g^2 & \rho_g(u_g-\lambda) & \alpha_g\rho_g & 0 \\ \alpha_l(u_g-\lambda)/c_l^2 & -\rho_l(u_l-\lambda) & 0 & \alpha_l\rho_l \\ \alpha_g & 0 & \alpha_g\rho_g(u_g-\lambda) & 0 \\ \alpha_l & 0 & 0 & \alpha_l\rho_l(u_l-\lambda) \end{vmatrix}=0 \tag{14}$$

It should be noted that the above equation is the characteristic equation for the system with matrices **A**, **B** and **F** (when written as Eq. (8)) as follows,

$$\mathbf{A}=\begin{bmatrix} \frac{\alpha_g}{c_g^2} & \rho_g & 0 & 0 \\ \frac{\alpha_l}{c_l^2} & -\rho_l & 0 & 0 \\ 0 & 0 & \alpha_g\rho_g & 0 \\ 0 & 0 & 0 & \alpha_l\rho_l \end{bmatrix} \tag{15}$$

$$\mathbf{B}=\begin{bmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g\rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l\rho_l \\ \alpha_g & 0 & \alpha_g\rho_g u_g & 0 \\ \alpha_l & 0 & 0 & \alpha_l\rho_l u_l \end{bmatrix} \tag{16}$$

and

$$\mathbf{F}=\begin{bmatrix} p & \alpha_g & u_g & u_l \end{bmatrix}^T \tag{17}$$

From the above matrices and Eqs. (2), (3) and (5), it can be inferred that this system represents the system of the equation for the isothermal two-phase flows (although source terms will be different for the isothermal case, they are irrelevant for the analysis of the characteristics). Further analysis is carried out only for the isothermal flows as eigenvalues corresponding to energy equations are already obtained.

2.1. Review of Characteristic Equation for Incompressible Isothermal Flow

A review of the eigenvalues of the incompressible isothermal two-fluid two phase model is given in this section. The derivations and results given in this section have been obtained previously [5, 6]. These results are reproduced here for the sake of completeness.

The characteristic equation of the incompressible case can be obtained by considering the case, $1/c_g^2 = 1/c_l^2 = 0$, Therefore, the characteristic equation can be written (Eq. (14)) as,

$$\begin{vmatrix} 0 & \rho_g(u_g-\lambda) & \alpha_g\rho_g & 0 \\ 0 & -\rho_l(u_l-\lambda) & 0 & \alpha_l\rho_l \\ \alpha_g & 0 & \alpha_g\rho_g(u_g-\lambda) & 0 \\ \alpha_l & 0 & 0 & \alpha_l\rho_l(u_l-\lambda) \end{vmatrix} = 0 \tag{18}$$

Expanding on the last row, the above equation yields,

$$\alpha_l\rho_g(u_g-\lambda)^2 + \alpha_g\rho_l(u_l-\lambda)^2 = 0 \tag{19}$$

On further simplification, the above equation yields,

$$(\alpha_l\rho_g+\alpha_g\rho_l)\lambda^2 - 2(u_g\alpha_l\rho_g + u_l\alpha_g\rho_l)\lambda + u_g^2\alpha_l\rho_g + u_l^2\alpha_g\rho_l = 0 \tag{20}$$

The solution of the characteristic equation is as follows:

$$\lambda = \frac{(u_g\alpha_l\rho_g + u_l\alpha_g\rho_l)}{(\alpha_l\rho_g+\alpha_g\rho_l)} \pm \frac{\sqrt{-(u_g-u_l)^2\alpha_g\rho_l\alpha_l\rho_g}}{(\alpha_l\rho_g+\alpha_g\rho_l)} \tag{21}$$

2.2. Compressibility Effect on the Characteristic Equation

For compressible two-phase flows the characteristic equation is a quartic equation and hence the eigenvalues can not be obtained analytically in a simple form. Characteristic equation (Eq. (14)) is expanded and separated into incompressible and compressible parts as follows,

$$\left[-\frac{\alpha_l\rho_g}{c_l^2}(u_g-\lambda)^2(u_l-\lambda)^2 - \frac{\alpha_g\rho_l}{c_g^2}(u_g-\lambda)^2(u_l-\lambda)^2\right] + \left[\alpha_l\rho_g(u_g-\lambda)^2 + \alpha_g\rho_l(u_l-\lambda)^2\right] = 0 \tag{22}$$

In the above equation these two parts are given in two separate square brackets. It should be noted that the first square bracket of the above characteristic equation is zero for the incompressible flow. The two square brackets in the subsequent equations also have the compressible and incompressible parts, respectively.
The above equation can be written as follows,

$$-\frac{(c_g^2\alpha_l\rho_g + c_l^2\alpha_g\rho_l)}{c_g^2 c_l^2}\left[(u_g-\lambda)^2(u_l-\lambda)^2\right] + \left[\alpha_l\rho_g(u_g-\lambda)^2 + \alpha_g\rho_l(u_l-\lambda)^2\right] = 0 \tag{23}$$

2.2.1 Galilean transformation

Substituting,

$$\lambda = Y + \frac{u_g + u_l}{2} \tag{24}$$

one gets,

$$-\frac{(c_g^2\alpha_l\rho_g + c_l^2\alpha_g\rho_l)}{c_g^2 c_l^2}\left[(\Delta u - Y)^2(\Delta u + Y)^2\right] + \left[\alpha_l\rho_g(\Delta u - Y)^2 + \alpha_g\rho_l(\Delta u + Y)^2\right] = 0 \tag{25}$$

where,

$$\Delta u = (u_g - u_l)/2\,. \tag{26}$$

The above substitution is equivalent to a Galilean transformation such that after the transformation one phase moves with the velocity Δu and the other phase with velocity $-\Delta u$. Dividing Eq. (25) by $\alpha_l\rho_g + \alpha_g\rho_l$, one gets,

$$-\frac{(c_g^2\alpha_l\rho_g + c_l^2\alpha_g\rho_l)}{c_g^2 c_l^2(\alpha_l\rho_g + \alpha_g\rho_l)}\left[(\Delta u^2 - Y^2)^2\right] + \left[\frac{\alpha_l\rho_g(\Delta u - Y)^2 + \alpha_g\rho_l(\Delta u + Y)^2}{(\alpha_l\rho_g + \alpha_g\rho_l)}\right] = 0\,. \tag{27}$$

Defining a sound speed, $c_{m0}^2 = \frac{c_g^2 c_l^2(\alpha_l\rho_g + \alpha_g\rho_l)}{(c_g^2\alpha_l\rho_g + c_l^2\alpha_g\rho_l)}$, the above equation can be written as,

$$-\frac{1}{c_{m0}^2}\left[(\Delta u^2 - Y^2)^2\right] + \frac{1}{(\alpha_l\rho_g + \alpha_g\rho_l)}\left[\alpha_l\rho_g(\Delta u - Y)^2 + \alpha_g\rho_l(\Delta u + Y)^2\right] = 0 \tag{28}$$

2.2.2. Eigenvalues for the zero slip velocity

In case $\Delta u = 0$ (i.e. $u_g = u_l$) the above equation becomes,

$$-\frac{Y^4}{c_{m0}^2} + Y^2 = 0 \tag{29}$$

The roots of the above equation can be written as

$$Y = \begin{cases} 0 \ \text{(repeated root)} \\ \pm c_{m0} \end{cases} \tag{30}$$

The corresponding solutions for the Eq. (22) are,

$$\lambda = \begin{cases} u_g \ \ \text{(repeated root)} \\ u_g \pm c_{m0} \end{cases} \tag{31}$$

2.2.3 Non-dimensional characteristic equation

For the case, $\Delta u \neq 0$, one can define, $Z = \Delta u Y$, to rewrite Eq. (28) as follows,

$$-\frac{\Delta u^4}{c_{m0}^2}\left[(1-Z^2)^2\right] + \frac{\Delta u^2}{(\alpha_l \rho_g + \alpha_g \rho_l)}\left[\alpha_l \rho_g (1-Z)^2 + \alpha_g \rho_l (1+Z)^2\right] = 0 \tag{32}$$

Dividing by Δu^2, the above equation becomes,

$$-\frac{\Delta u^2}{c_{m0}^2}\left[(1-Z^2)^2\right] + \frac{1}{(\alpha_l \rho_g + \alpha_g \rho_l)}\left[\alpha_l \rho_g (1-Z)^2 + \alpha_g \rho_l (1+Z)^2\right] = 0 \tag{33}$$

Rearranging the terms in the above equation, one gets,

$$-\frac{\Delta u^2}{c_{m0}^2}\left[(1-Z^2)^2\right] + \left[Z^2 - 2\frac{\alpha_l \rho_g - \alpha_g \rho_l}{(\alpha_l \rho_g + \alpha_g \rho_l)} Z + 1\right] = 0 \tag{34}$$

Defining the slip Mach number, M_s, as,

$$\frac{\Delta u^2}{c_{m0}^2} = M_s^2, \tag{35}$$

and

$$r = \frac{\alpha_l \rho_g - \alpha_g \rho_l}{\alpha_l \rho_g + \alpha_g \rho_l} \tag{36}$$

$$M_s^2\left[-(1-Z^2)^2\right] + \left[Z^2 - 2rZ + 1\right] = 0 \tag{37}$$

It should be noted that $r^2 \le 1$. Also, $r^2 = 1$ when either α_g or α_l is zero.

From Eq. (49) it is clear that for the small slip velocity case i.e. $M_s^2 << 1$, the effect of compressibility on the characteristics of the system can be ignored. It needs to be emphasized that $c_g^2 \le c_{m0}^2 \le c_l^2$. Therefore, in the applications in which $\Delta u^2 << c_g^2$, the abovementioned compressibility effect can be ignored irrespective of the void fraction.

3. SERIES SOLUTION OF THE CHARACTERISTICS EQUATION

An analytical solution of the characteristic equation, which is a quartic polynomial equation, can be obtained. However, this solution is in the radical form and not very useful for any physical interpretation. As discussed earlier, for most of the nuclear applications values of M_s are small. The roots of the characteristic equation can be found as a power series expansion about the point $M_s = 0$. These roots are useful for obtaining choke flow criterion.

For $M_s = 0$, the two roots of the characteristic equation, Eq. (37), are as follows,

$$Z = r \pm \sqrt{r^2 - 1} \tag{38}$$

These two roots are complex, except for $r^2 = 1$. (Note that $r^2 \le 1$ as discussed before). Since these roots are obtained by neglecting the compressible part of the equation these are same as those obtained for the incompressible flow. The corresponding values of the λ (obtained using Eqs. (24), (26) and (21)) are same as that for the incompressible flow given in the Eq. (22).

A numerical solution can be obtained for Eq. (37) for given values of M_s and *r*. From the roots obtained (using Mathematica®) for small values of M_s, it is observed, that two of the roots approach the values given by Eq. (38) as $M_s \to 0$. The remaining two roots are real and approach the following values,

$$Z = -r \pm \frac{1}{M_s} \tag{39}$$

The corresponding values of the λ can be obtained as (from Eqs. (24), (26) and (36)),

$$\lambda = \frac{(u_g \alpha_g \rho_l + u_l \alpha_l \rho_g)}{(\alpha_l \rho_g + \alpha_g \rho_l)} \pm c_{m0} \tag{40}$$

Defining,

$$u_{m0} = \frac{(u_g \alpha_g \rho_l + u_l \alpha_l \rho_g)}{(\alpha_l \rho_g + \alpha_g \rho_l)} \tag{41}$$

the eigenvalues can be written as,

$$\lambda = u_{m0} \pm c_{m0} \tag{42}$$

in a form similar to single phase eigenvalues.

It is interesting to note that, for $\alpha_g \to 0$ the above value reduces to $\lambda = u_l \pm c_l$ and for, $\alpha_l \to 0$ one can obtain that $\lambda = u_g \pm c_g$. These values are the eigenvalues for the single phase flows. Moreover, for $u_g \to u_l$, the above roots reduce to the roots for the zero slip velocity case (Eq.(30)). It is also important to note that the four eigenvalues obtained by Trapp and Ransom (1982) [1] are same as those given in Eqs. (21) and (40) if virtual mass and thermal non-equilibrium terms are ignored.

The difference between the roots obtained numerically and those obtained from Eq. (39) is plotted in Fig. 1. From the plot it can be seen that error is $O(M_s)$. However, the residual R of the non-dimensional characteristic equation, defined as follows

$$R(Z) \equiv M_s^2[-(1-Z^2)^2] + \left[Z^2 - 2rZ + 1\right] \tag{43}$$

has the following value

$$R\left(-r \pm \frac{1}{M_s}\right) = \left(r^2 - 1\right)\left(-3 \pm 4M_s r + (1-r^2)M_s^2\right) \tag{44}$$

which does not go to zero as $M_s \to 0$.

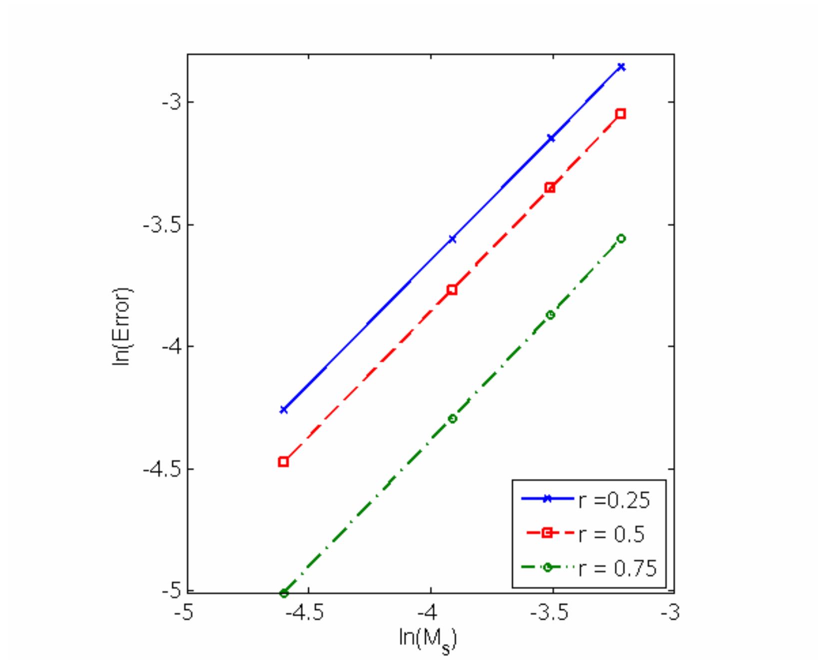

Fig. 1. The plot of error in the eigenvalue given by Eq. (39) as a function of M_s. The plot shows that error is $O(M_s)$.

In order to find higher order terms in the root of the characteristic equation, one can substitute $Z = -r \pm \frac{1}{M_s} + f_1(r)M_s$ in the non-dimensional characteristic equation to yield,

$$R\left(-r \pm \frac{1}{M_s} + f_1(r)M_s\right) = 3\left(1-r^2\right) \mp 2f_1(r) + M_s r\left(\mp 4(1-r^2) + 8f_1(r)\right) + O(M_s^2) \qquad (45)$$

From the above equation it can be seen that choosing,

$$f_1(r) = \pm \frac{3}{2}\left(1-r^2\right) \qquad (46)$$

will make the residual to be $O(M_s)$. The difference in the root obtained with

$$Z = -r \pm \frac{1}{M_s} \pm \frac{3}{2}\left(1 - r^2\right)M_s \tag{47}$$

and the numerical solution is plotted in the Fig. 2. It can be seen that the error is $O(M_s^2)$. It is emphasized here that for the approximate root given in the Eq. (47) the error is $O(M_s^2)$ while the residual is $O(M_s)$. The behavior is similar to the root given by Eq. (39) in which case root has the error $O(M_s)$ while the residual is $O(1)$ (Eq. (44)). Therefore, it is seen that the error in the root is one order lower than the residual.

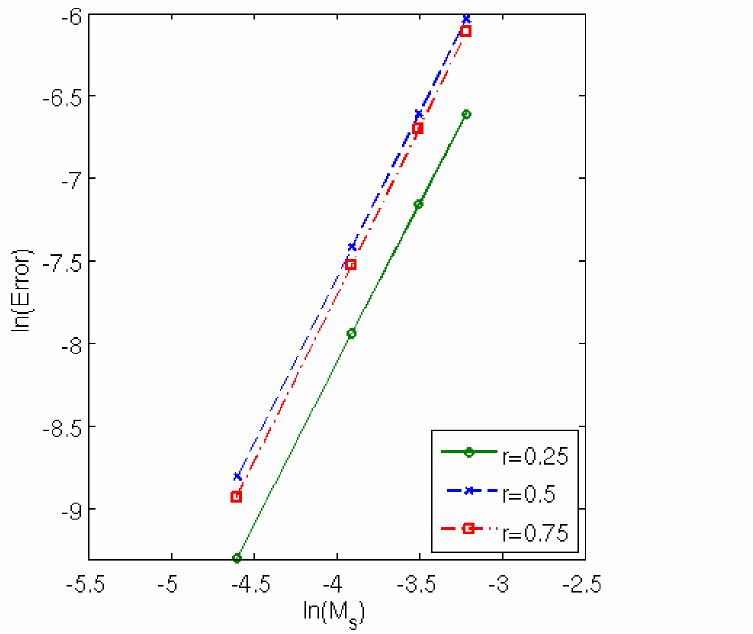

Fig. 2. The plot of error in the eigenvalue given by Eq. (47) as a function of M_s. The plot shows that error is $O(M_s^2)$.

To obtain the next term in the series one can evaluate residual with

$$Z = -r \pm \frac{1}{M_s} \pm \frac{3}{2}\left(1 - r^2\right)M_s + f_2(r)M_s^2 \tag{48}$$

and then evaluate $f_2(r)$ such that $O(M_s)$ term in the residual is eliminated. In fact, the roots can be found with arbitrary accuracy (as long as $M_s < 1$), by using the above process recursively. The first few terms are obtained for the two real roots are as follows,

$$\begin{aligned} Z = -r \pm \frac{1}{M_s} \pm \frac{3}{2}(1-r^2)M_s + 4r(1-r^2)M_s^2 \pm \frac{5}{8}(1-r^2)(21r^2-5)M_s^3 \\ + 24r(1-r^2)(2r^2-1)M_s^4 + O(M_s^5) \end{aligned} \tag{49}$$

3.1. Analysis of the Real Roots of the Characteristic Equation

It can be seen that the third and fifth term in the series have the same sign as the $1/M_s$ while the remaining terms do not change sign. Moreover, fourth and sixth term have r as a factor. In the light of the above observation, the terms in the Eq. (49) are rearranged as follows,

$$\begin{aligned} Z = -r\left\{1 - 4(1-r^2)M_s^2 - 24(1-r^2)(2r^2-1)M_s^4 + O(M_s^6)\right\} \\ \pm \frac{1}{M_s}\left\{1 + \frac{3}{2}(1-r^2)M_s^2 + \frac{5}{8}(1-r^2)(21r^2-5)M_s^4 + O(M_s^6)\right\} \end{aligned} \tag{50}$$

In the above equation, terms in Eq. (49) are written as two different series. As $M_s \to 0$, the first series goes to $-r$ while the second series approaches $1/M_s$.

Using equation (24) and (26), the values of the λ are obtained corresponding to the Z values in the Eq. (50) as follows,

$$\begin{aligned} \lambda = u_{m0} - r(\Delta u)\left\{-4(1-r^2)M_s^2 - 24(1-r^2)(2r^2-1)M_s^4 + O(M_s^6)\right\} \\ \pm c_{m0}\left\{1 + \frac{3}{2}(1-r^2)M_s^2 + \frac{5}{8}(1-r^2)(21r^2-5)M_s^4 + O(M_s^6)\right\} \end{aligned} \tag{51}$$

The definitions of the u_{m0} and c_{m0} used in the above equation are given in Eq. (41) and Eq. (28), respectively. The Eq. (51) can be written as,

$$\lambda = u_m \pm c_m \tag{52}$$

where,

$$u_m \equiv u_{m0} + r(\Delta u)\left\{4(1-r^2)M_s^2 + 24(1-r^2)(2r^2-1)M_s^4 + O(M_s^6)\right\} \quad (53)$$

and

$$c_m = c_{m0}\left\{1 + \frac{3}{2}(1-r^2)M_s^2 + \frac{5}{8}(1-r^2)(21r^2-5)M_s^4 + O(M_s^6)\right\} \quad (54)$$

Note that as $\alpha_g \to 0$, $r^2 \to 1$, $u_{m0} \to u_l$ and $c_{m0} \to c_l$. Therefore, the real eigenvalues approach $\lambda = u_l \pm c_l$. (Here, it is assumed that higher order terms also have a factor of $r^2 - 1$ in them). Similar result can be obtained for $\alpha_l \to 0$. Clearly, the roots approach the single phase eigenvalues in the abovementioned limits. Also, in case $u_g = u_l$, one gets, $u_{m0} = u_g = u_l$ and $M_s = 0$. Therefore, the two eigenvalues are given as, $\lambda = u_g \pm c_{m0}$ same as those obtained earlier (Eq. (31)).

3.2. Complex Roots of the Characteristic Equation

The two complex conjugate roots in the series form can be found in a similar manner as the real roots. The complex roots of the non-dimensional characteristic equation are as follows,

$$\begin{aligned} Z = {} & r\left\{1 - 4(1-r^2)M_s^2 - 24(1-r^2)(2r^2-1)M_s^4 + O(M_s^6)\right\} \\ & \pm\sqrt{r^2-1}\left\{1 + (-2+4r^2)M_s^2 + 6(1-8r^2+r^4)M_s^4 + O(M_s^6)\right\} \end{aligned} \quad (55)$$

It is interesting to note that, the series given in the first bracket in the equation is same as the series in the first bracket in the Eq. (50). The corresponding value of the λ is as follows,

$$\begin{aligned} \lambda = {} & \frac{(u_g\alpha_l\rho_g + u_l\alpha_g\rho_l)}{(\alpha_l\rho_g + \alpha_g\rho_l)} + r\Delta u\left\{4(1-r^2)M_s^2 + 24(1-r^2)(2r^2-1)M_s^4 + O(M_s^6)\right\} \\ & \pm i\frac{\sqrt{(u_g-u_l)^2\alpha_g\rho_l\alpha_l\rho_g}}{(\alpha_l\rho_g + \alpha_g\rho_l)}\left\{1 + (-2+4r^2)M_s^2 + 6(1-8r^2+r^4)M_s^4 + O(M_s^6)\right\} \end{aligned} \quad (56)$$

From the above equation it is obvious that the real part of the above eigenvalues is positive if u_g and u_l are positive

4. CHOKE FLOW CRITERION

A choked condition exists when mass flow rate becomes independent of the downstream conditions. In other words, no information can propagate in the upstream direction under this condition. The real part of the eigenvalue represents velocity of the signal propagation and the imaginary part is the growth (or decay) rate of that signal. Therefore, if the real part of all the eigenvalues is positive then no signal propagates in the upstream direction (choosing downstream direction to be the positive direction). From the Eq. (52), two of the eigenvalues are positive if $u_m \geq c_m$. Remaining four eigenvalues (two associated with energy equations and two complex eigenvalues) have positive real parts if u_g and u_l are positive, as discussed before. Therefore, choke flow exists when $u_m \geq c_m$, $u_g \geq 0$ and $u_l \geq 0$.

5. CONCLUSIONS

A non-dimensional characteristic equation is obtained for the two-phase two-fluid model. The equation is obtained in terms of a slip Mach number (M_s). The eigenvalues of the characteristics equation of this model have been obtained as power series expansion about the point $M_s = 0$. The choking condition exists when all the eigenvalues are positive. The abovementioned series solution is used to obtain a choking criterion for the two-phase flow.

ACKNOWLEDGMENTS

This work has been carried out for the U.S. Department of Energy Office of Nuclear Energy under DOE Idaho Operations Office Contract DE-AC07-05ID14517 (INL/CON-09-15438).

REFERENCES

1. J. A. Trapp, V.H. Ransom, “A Choked Flow Calculation Criterion for Non-Homogeneous, Non-Equilibrium, Two-phase Flows”, *Int. J. Multiphase Flow*, **8(6)**, pp. 669-681 (1982).
2. U. S. N. R. C., *RELAP 5/MOD 3.3 Code Manual Volume I*, NUREG/CR-5535 Edition (December 2001).
3. M. S. Chung, K. S. Chang, S. J. Lee, “Wave Propagation in Two-Phase Flow Based on a New Hyperbolic Two-Fluid Model”, *Numerical Heat Transfer Part A*, **38**, pp. 169-191, (2000).
4. M. S. Chung, W. J. Lee, K. S. Ha, “Choked Flow Calculation of Two-Phase Bubbly Flow”, *Numerical Heat Transfer Part A*, **42**, pp. 297-305 (2002).
5. J. D. Ramshaw, J. A. Trapp, “Charesteristics, Stability, and Short-Wavelength Phenomena in Two-Phase Flow Equation Systems,” *Nucl. Sci. Eng.*, **66**, pp. 93-102 (1978).
6. J. H. Stuhmiller, “The Influence of Interfacial Pressure on the Character of the Two-phase Flow Model Equations,” *Int. J. Multiphase Flow*, **3**, pp. 551-560 (1977).