International Conference on Mathematics, Computational Methods & Reactor Physics (M&C 2009)
Saratoga Springs, New York, May 3-7, 2009, on CD-ROM, American Nuclear Society, LaGrange Park, IL (2009)

MARKOVIAN APPROACH: FROM ISING MODEL TO STOCHASTIC RADIATIVE TRANSFER

Evgueni Kassianov
Pacific Northwest National Laboratory
Richland, WA, 99352, USA
Evgueni.Kassianov@pnl.gov

Dana Veron
University of Delaware
Newark, DE, 19716, USA
dveron@cms.udel.edu

ABSTRACT

The origin of the Markovian approach can be traced back to 1906; however, it gained explicit recognition in the last few decades. This overview outlines some important applications of the Markovian approach, which illustrate its immense prestige, respect, and success. These applications include examples in the statistical physics, astronomy, mathematics, computational science and the stochastic transport problem. In particular, the overview highlights important contributions made by Pomraning and Titov to the neutron and radiation transport theory in a stochastic medium with homogeneous statistics. Using simple probabilistic assumptions (Markovian approximation), they have introduced a simplified, but quite realistic, representation of the neutron/radiation transfer through a two-component discrete stochastic mixture. New concepts and methodologies introduced by these two distinguished scientists allow us to generalize the Markovian treatment to the stochastic medium with inhomogeneous statistics and demonstrate its improved predictive performance for the downwelling shortwave fluxes.

Key Words: Markovian statistics, stochastic radiative transfer, broken clouds, inhomogeneous model, analytical averaging

1. INTRODUCTION

The 20th century opened auspiciously with a significant achievement in probability theory. In a sequence of highly celebrated papers, starting in 1906, gifted Russian mathematician, A.A. Markov, generalized various limit laws established for independent random variables (e.g., the law of large numbers) to dependent ones [1]. The generalization allows a departure from independence by accepting the dependence on the most recent available information. In other words, the best prediction of a future event depends on what happens “today,” and any information from “past” events is irrelevant in such predictions. In the course of his work A.A. Markov so advanced the theory of dependent variables, now called Markov chains, that it became applicable to many other areas. Here we outline some important applications of the Markovian approach and highlight related accomplishments.

The theory behind the Metropolis-Hastings algorithm relies on the properties of Markov chains. Since the Metropolis-Hastings algorithm is extremely versatile and allows one to simulate desired complex multivariable distributions, it has been used extensively in physics and computational science [2]. The methods based on simulating a Markov chain are applied widely for solving a system of simultaneous linear equations and integral equations [3].

Other examples are related to the Markov random fields (MRFs), which can be considered a logical extension of Markov chains on spatially distributed random variables. The Markovian nature of nearest-neighbor interactions was tackled by one of the greatest mathematical physicist of the 20th century, C. Chandrasekhar. In his paper "*Stochastic Problems in Physics and Astronomy*," published in 1942 and cited more than 1500 times, Chandrasekhar applied the Markovian approach to derive the statistics of the gravitational field arising from a random distribution of stars [4]. Specifically, it was assumed that the force acting on a star depends on its *neighboring* stars and their spatial distribution is subject to well-defined laws of fluctuations.

Under certain conditions, the MRFs are equivalent to Gibbs random fields (GRFs), which have been studied extensively in statistical physics during the last several decades. For example, the Ising model [5], originally suggested in 1920, belongs to the family of the GRFs and is identical to the binary MRFs. The equivalence between MRFs and GRFs was provided by the Hammersley-Clifford theorem [6], thus probabilistic tools developed for the GRFs can be applied for MRFs as the need arises (and vice versa).

Pomraning and his colleagues successfully applied binary MRFs in various kinetic theory models, which have described the particle transport through a two-component (liquid water and vapor) stochastic mixture with Markovian statistics [7]. Later, they generalized their approach to radiative transfer through broken clouds and illustrated [8] that the integrated equation approach introduced by Titov for a Markovian mixture [9] is equivalent to Pomraning's differential model developed in kinetic theory. Below we discuss an application of the Markovian approach for radiative transfer (RT) in random media.

2. STOCHASTIC RADIATIVE TRANSFER

Let us consider the radiative transfer equation (RTE). The history of the RTE, and its important practical applications have been discussed appropriately [10-13, and references therein]. In particular, it was illustrated that the standard RTE can be derived successfully from the Maxwell equations and its parameters, such as the extinction and phase matrices, have explicit physical meaning [14]. In last few decades, much attention is given to the stochastic RTE [15, 16, and references therein], which can be considered as the transport equation with random parameters

$$\boldsymbol{\omega}\nabla I(\mathbf{r},\boldsymbol{\omega})+\sigma(\mathbf{r})I(\mathbf{r},\boldsymbol{\omega})=\sigma_S(\mathbf{r})\int_{4\pi} g(\boldsymbol{\omega},\boldsymbol{\omega}')I(\mathbf{r},\boldsymbol{\omega},\boldsymbol{\omega}')d\boldsymbol{\omega}'+S(\mathbf{r},\boldsymbol{\omega}), \tag{1}$$

here $I(\mathbf{r},\boldsymbol{\omega})$ is intensity of radiation at the point $\mathbf{r}=(x,y,z)$ in direction $\boldsymbol{\omega}=(a,b,c)$, $S(\mathbf{r},\boldsymbol{\omega})$ is the external source, $\sigma(\mathbf{r})$, $\sigma_S(\mathbf{r})$ and $g(\mathbf{r},\boldsymbol{\omega},\boldsymbol{\omega}')$ are the extinction coefficient (total cross section), the scattering coefficient (scattering cross section), and the scattering phase function, respectively. Optical properties, σ and σ_s, and the external source S are assumed to be random variables with known statistical properties.

To obtain the mean intensity, one needs to apply a numerical or analytical averaging of the stochastic transfer equation (1). There are three main steps in the numerical averaging process: (a) the simulation of a large number of realizations of the inhomogeneous medium with a given distribution of $\sigma(\mathbf{r})$ and $\sigma_s(\mathbf{r})$, (b) solving the deterministic radiative transfer equation in each of the realizations by using well-known techniques [11], and (c) obtaining the statistics of the intensity after appropriate processing. Several methods have been developed to generate such realizations (Fig. 1) with given or observed statistical properties [17-20]. Typically, these methods incorporate one- and two-point statistics. By applying the numerical averaging, one can obtain the desired statistics of the intensity and other related radiative properties (e.g., mean radiative fluxes). However, performing the 3D RT calculations for a large number of cloud realizations is computationally expensive.

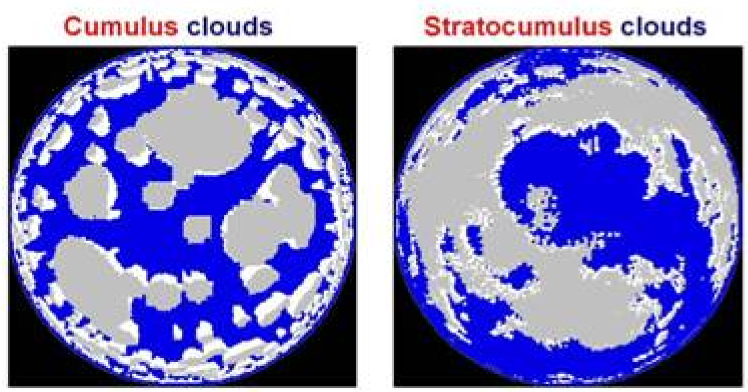

Figure.1. Computer realizations of hemispherical images generated by the stochastic models [21]: cumulus (left) and stratocumulus (right) clouds; cloud bases (gray), sides (white).

Since the numerical averaging requires significant computer time, it is advisable to derive approximate solutions by using analytical averaging. Such solutions link statistics of optical properties and radiation. To illustrate that, we consider the obtaining of the ensemble-averaged intensity of unscattered (direct) radiation $\langle T_d(\mathbf{r},\boldsymbol{\omega})\rangle$ in the source-free case (S=0). Here and below we will use angular brackets for the ensemble averages. The corresponding equation represents the simplest case of a purely absorbing (no scattering, or $\sigma_s(\mathbf{r})$=0) random medium

$$\boldsymbol{\omega}\nabla T_d(\mathbf{r},\boldsymbol{\omega})+\sigma(\mathbf{r})T_d(\mathbf{r},\boldsymbol{\omega})=0 \tag{2a}$$

If the boundary condition is

$$T_d(\mathbf{r}_0,\boldsymbol{\omega}) = T_d(x,y,0,\boldsymbol{\omega}) = 1; \quad c>0\,, \tag{2b}$$

then we have the solution of Equations (2a) and (2b)

$$T_d(\mathbf{r},\boldsymbol{\omega}) = \exp[-\tau(\mathbf{r},\mathbf{r}_0)], \tag{3a}$$

where $\tau(\mathbf{r},\mathbf{r}_0)$ is the random optical depth

$$\tau(\mathbf{r},\mathbf{r}_0) = \frac{1}{c}\int_0^z \sigma(\mathbf{r}')d\xi\,, \quad \mathbf{r}' = \mathbf{r} + \boldsymbol{\omega}\times(\xi - z)/c \tag{3b}$$

Equation (3a) represents well-known Beer's law of exponential direct transmittance. Since optical depth τ is random, the direct transmittance T_d is random as well.

Here we outline two different methods of obtaining the ensemble-averaged direct transmittance $\langle T_d \rangle$ [7, 22]. The first method involves the ensemble averaging of exponential attenuation (3a) where analytical integration of solution (3a) should be weighted by the corresponding probability distribution function $f(\tau)$. In the case of spatially *uniform* extinction coefficient $\sigma(\mathbf{r}) = \sigma$, analytical expressions of $\langle T_d \rangle$ can obtained quite easily for many commonly-used probability density functions $f(\sigma)$ [23]. Note that this approach also makes it possible to derive the distribution functions of radiative fluxes as is done by Stephens et al. [24]. When σ fluctuations are spatially correlated, the $\langle T_d \rangle$ derivation becomes much more complicated [25, 26] and analytical expressions of $\langle T_d \rangle$ can be obtained only for a limited number of statistical models, such as models with Gaussian [25, 27] and Markovian [7, 9] statistics.

The second method involves ensemble averaging of Equation (2a). To perform such averaging, let us represent the extinction coefficient σ and transmittance T_d as $\sigma = \langle \sigma \rangle + \tilde{\sigma}$ and $T_d = \langle T_d \rangle + \tilde{T}_d$, where $\tilde{\sigma}$ and $\tilde{T}_d$ are the corresponding fluctuating quantities with zero expected values ($\langle \tilde{\sigma} \rangle = \langle \tilde{T}_d \rangle$ =0). With such representation, the ensemble averaging of Equation (2a) gives

$$\boldsymbol{\omega}\nabla\langle T_d \rangle + \langle \sigma \rangle \langle T_d \rangle + \langle \tilde{\sigma}\tilde{T}_d \rangle = 0 \tag{4}$$

Equation (4) contains unknown term $\langle \tilde{\sigma}\,\tilde{T}_d \rangle$. This term defines the statistical correction to the transport description and can be described by infinite series with multipoint spatial correlations $\langle \tilde{\sigma}(\mathbf{r}_1)\tilde{\sigma}(\mathbf{r}_2)\cdots\tilde{\sigma}(\mathbf{r}_{n-1})\rangle$ [7]. In practice, only lower-order statistical moments of $\tilde{\sigma}$ are available. Thus, for practical applications these infinite series are typically truncated at a specified low order. Such truncation is equivalent to the closure procedures [22, 28]. Complex statistical correlation terms in these infinite series become simpler when $\tilde{\sigma}$ fluctuations are approximated by some popular statistical models [7, 25, 27].

Under the Markovian model [7], the considered two ways provide the same *explicit* solution

$$\langle T_d(s)\rangle = \sum_{i=1}^{2} C_i \exp\{-\chi_i s\}, \tag{5}$$

where coefficients C_i and χ_i, $i = 1,2$ are functions of Markovian model parameters. In the presence of scattering the Markovian approach allows one to obtain an *approximate* solution in the form of two coupled equations [7], which are known as Levermore-Pomraning model,

$$\boldsymbol{\omega}\nabla(p_i I_i) + \sigma_i p_i I_i = \sigma_{si} \int_{4\pi} g_i(\boldsymbol{\omega},\boldsymbol{\omega}') p_i I_i(\boldsymbol{\omega}') d\boldsymbol{\omega}' + p_i S_i + \frac{p_j I_j}{\lambda_j} - \frac{p_i I_i}{\lambda_i} \ , \tag{6}$$

where $i = 0,1$, $j \neq i$, and $p_i(\mathbf{r})$ is the probability of component i of the mixture being at position $\mathbf{r}$, and the $I_i(\mathbf{r},\boldsymbol{\omega})$ is conditional ensemble-averaged intensity conditioned upon position $\mathbf{r}$ being in component i. The unconditional ensemble-averaged intensity is defined by

$$\langle I\rangle = \sum_{i=1}^{2} p_i I_i \tag{7}$$

This approach has been successfully applied to study neutron transport through randomly distributed lumps and absorbers [29, 30] and radiative transfer through random media with Rayleigh scattering [31] and clouds [32, and references therein].

Titov has provided an alternative derivation of two coupled integral equations of the ensemble-averaged intensity [9]

$$\langle I(s)\rangle + \int_0^s C_1 p_1(s') I_1(s') ds' = F \tag{8}$$

$$p_1(s) I_1(s) + \int_0^s P_{11}(s',s) C_1 p_1(s') I_1(s') ds' = p_1 F \ , \tag{9}$$

where the collision operator is defined by

$$C_1(p_1 I_1) = \sigma_1 p_1 I_1 - \sigma_{s1} \int_{4\pi} g_1(\boldsymbol{\omega},\boldsymbol{\omega}') p_1 I_1(\boldsymbol{\omega}') d\boldsymbol{\omega}' \tag{10}$$

and F denotes the incoming intensity at this point, $P_{ij}(s', s)$ is the conditional probability that position s is in mixture component j, given that position s' is in component i, in Equation (9) $i = j = 1$. In other words, the conditional probability of the cloud presence $P_{11}(s', s)$ is the conditional probability of the event (point s is in cloud), given that another event (point s' is in cloud) has occurred.

Coupled integral equations, that are similar to (8)-(10), were obtained early by Avaste and Vainikko [33] with aim to describe radiative transfer through single-layer broken clouds. Throughout more than three decades, other researchers continued to create more powerful mathematical machinery and employ it successfully in the further investigation of complex cloud-radiation [34, 35] and vegetation-radiation [36] interactions. These achievements include (1) a set of equations that link the cloud/vegetation statistics (e.g., the unconditional and conditional probabilities of the cloud presence) with the ensemble-averaged radiative characteristics and (2) mathematically rigorous methods for their solution. In particular, the Markovian approach has been applied successfully to study the influence of the stochastic geometry of broken clouds on solar [34] and thermal [37] radiation, and spatially confined beams from artificial radiation sources [38].

The considered Markovian models assume that both the cloud size and the cloud spacing are exponential. Is that Markovian assumption realistic? To describe size distribution of clouds, a power law [39, 40] has been typically applied. However, for boundary layer convective clouds [41-43], such as fair weather cumulus and cumulus humilis, the cloud size distributions can be approximated quite accurately by exponential fits (Fig. 2). Thus, the considered Markovian models should be appropriate for such clouds. The observed cloud size distributions are applied to evaluate the predictions of stochastic (Markovian) RT model [32]: in comparison with a traditional plane-parallel RT model, the stochastic one does provide a significant improvement in calculations of the downwelling shortwave surface flux.

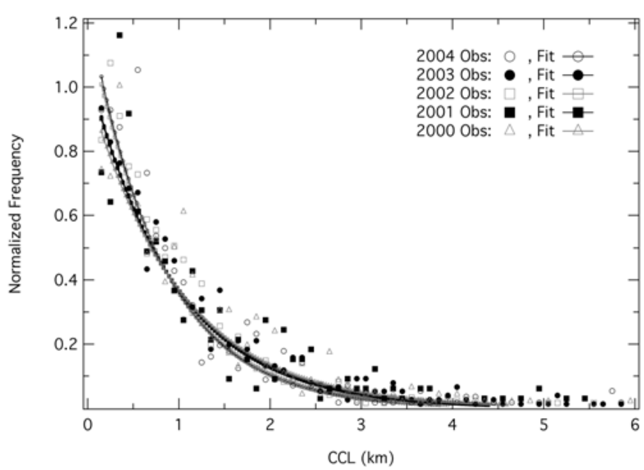

Figure 2. Distributions of cloud chord lengths [43] obtained in Oklahoma, USA for five summers (2000-2004). Lines correspond to the best-fit exponential distribution for each year.

These important results have been accomplished for the statistically homogeneous MRFs. For such fields, the unconditional probability of the cloud presence p_1 does not depend on the vertical coordinate, and the conditional one $P_{11}(s', s)$ depends only on the difference between two points for a fixed direction. As a result, these statistics are unable to describe the vertical variability of clouds. However, the substantial horizontal changes of cloud properties can be accompanied with their strong vertical variability. Thus, the extension of the Markovian approach to random fields with inhomogeneous statistics is highly desirable.

3. CLOUDS WITH INHOMOGENEOUS STATISTICS

Real clouds have strong horizontal and vertical variability. To extend the Markovian approach to statistically inhomogeneous broken clouds, one has to specify a statistical relationship between cloud layers. It was demonstrated that (i) the overlap of clouds at two levels tends to fall rapidly as their vertical separation is increased and (ii) the degree of overlap as a function of level separation can be described by a simple inverse-exponential expression [44, 45]. Thus, we assume that the statistical relationship between two adjusted layers is described by inverse-exponential expression.

We have performed such an extension by introducing a new statistically *inhomogeneous* Markovian model, which represents broken clouds as a set of correlated cloud layers [46]. Each layer is assumed to be homogeneous in the vertical but inhomogeneous in horizontal dimensions. For this model, the unconditional probability of the cloud presence p_1 can vary strongly with altitude, and the conditional one $P_{11}(s', s)$ is a function of the relative positions of two points s' and s. As an example, (1) if the points s' and s belong to the same *k*th layer, then

$$P_{11}(s', s) = \exp\left(-A_k |s' - s|\right)\left(1 - p_{1k}\right) + p_{1k} \quad ; \qquad (11a)$$

(2) if the points s' and s belong to different adjacent layers, namely *k*th and *m*th layers, then

$$P_{11}(s', s) = \exp\left(-A_{km} |s^* - s|\right)\left(P_{11}(s', s^*) - p_{1m}\right) + p_{1m} \quad , \qquad (11b)$$

where the parameter A_{km} determines the statistical relationship between *k*th and *m*th layers. Note, all these parameters A_k, A_{km} depend on both the 3D cloud structure and the positions of points s' and s. By changing the values of these parameters, one can describe different combinations of maximum and random cloud overlaps [46]. It was assumed also that for each *k*th layer, the domain-averaged optical properties are constant (piecewise constant approximation): the extinction $\sigma(\mathbf{r}) = \sigma(z) = \sigma_k$ and scattering $\sigma_s(\mathbf{r}) = \sigma_s(z) = \sigma_{sk}$ coefficients, and the scattering phase function $g(\mathbf{r}, \boldsymbol{\omega}, \boldsymbol{\omega}') = g(z, \boldsymbol{\omega}, \boldsymbol{\omega}') = g_k(\boldsymbol{\omega}, \boldsymbol{\omega}')$.

Using Equations (11a,b) and the piecewise constant approximation of optical properties Equations (8)-(10) can be re-written as

$$\langle I(s)\rangle = \int_0^s \frac{\sigma_{s1}(s')}{\sigma_1(s')}\phi(s,s')\,ds' \int_{4\pi} g_1(s',\boldsymbol{\omega},\boldsymbol{\omega}')\, f(s')\, d\boldsymbol{\omega}' + \langle T_d(s)\rangle F(0)\delta(\boldsymbol{\omega}-\boldsymbol{\omega}_{\oplus}), \tag{12}$$

where $f(s) = \sigma_1(s)p_1(s)I(s)$ is the mean collision density, $\delta(\cdot)$ is Dirac's delta function.

$$f(\mathbf{x}) = \int_X k(\mathbf{x},\mathbf{x}')\, f(\mathbf{x}')\, d\mathbf{x}' + \Psi(\mathbf{x}) \tag{13}$$

$$k(\mathbf{x},\mathbf{x}') = \frac{\sigma_{s1}(z)\, g_1(z,\boldsymbol{\omega},\boldsymbol{\omega}')\,\eta(\mathbf{r},\mathbf{r}')}{\sigma_1(z)2\pi\, |\mathbf{r}-\mathbf{r}'|^2}\,\delta\!\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} - \boldsymbol{\omega}\right) \tag{14}$$

where X is the phase space of coordinates and directions, $\mathbf{x} = (\mathbf{r}, \boldsymbol{\omega})$. All functions ϕ, η and ν are defined by the recurrent expressions [46].

There are three attractive features of the suggested treatment using the statistically inhomogeneous model. The first important feature is the model flexibility. The suggested model can describe the different combinations of random and maximum cloud overlaps normally used in general circulation models. Second is the ability to obtain the relatively few input parameters for this model from observations. This allows one to make a correct comparison of theory against experiment. Finally, the statistically inhomogeneous model is a logical extension of the statistically homogeneous models (single-layer broken clouds). Thus, a beautiful theory and elegant numerical methods, which were developed for statistically homogeneous models, can be used as the basis for solving the problem of solar radiative transfer in statistically inhomogeneous broken clouds.

We have demonstrated [46] that in extreme cases the equations obtained for the mean direct radiance $\langle T_d \rangle$ agree with corresponding equations previously derived for (i) statistically homogeneous broken clouds and (ii) vertically inhomogeneous overcast clouds. Also, we have estimated [47, 48] the accuracy and robustness of the equations (12)-(14) for diffuse radiance by comparing the mean radiative properties obtained by two independent methods. The first method (numerical averaging) provides a *reference* case. For a given 3D cloud field we calculated radiative properties. The ensemble- and domain-averaged radiative properties were obtained after appropriate processing. Since the full 3D cloud geometry is used in the radiative calculations, the calculated mean radiative properties are considered the reference values. The second method (analytical averaging) provides an *approximate* case. Approximated equations for the mean radiance, which have been derived by analytically averaging the stochastic radiative transfer equation, are also used for estimating ensemble-averaged radiative properties. Since only the bulk cloud statistics are used in the radiative calculations, the mean radiative properties obtained by this method are considered as approximations of the true radiative properties.

The approximated equations provide reasonable accuracy (~15%) for both the ensemble-averaged and domain-averaged radiative properties. Also, the angular distribution histograms

(Fig. 3) and the photon path length distributions of the mean albedo and transmittance, which have been obtained by exact and approximated methods, agree qualitatively and quantitatively.

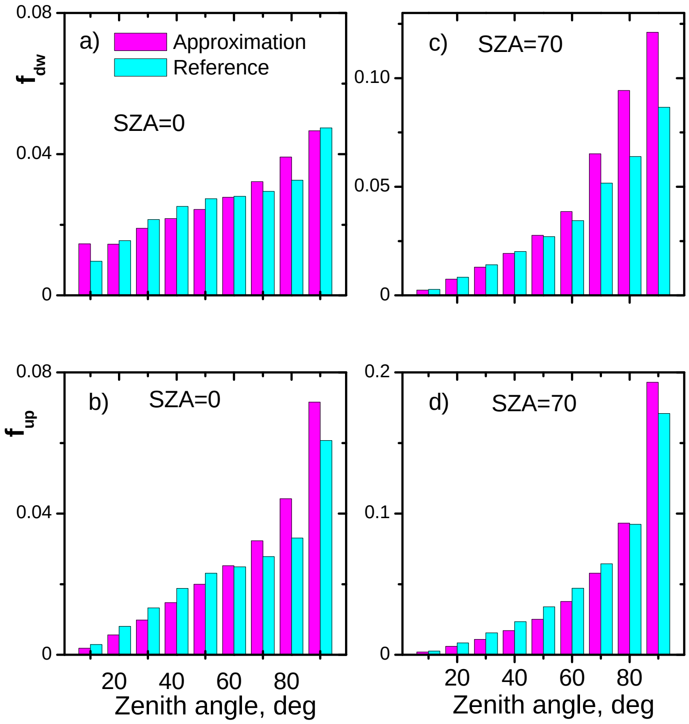

Figure 3. The mean angular distribution histograms of the transmitted (a,c) and reflected (b,d) radiation. These histograms [47] were obtained for two values of solar zenith angle (SZA) by using full 3D cloud structure (reference) and the bulk cloud statistics (approximation).

4. CONCLUSIONS

The historical application of the Markovian approach can be considered a 100-year adventure from a purely unpractical mathematical problem (dependent random variables) to a powerful tool with profound practical significance and consequences, which permeate many research areas and disciplines. The highlighted important achievements of its radiative transfer applications, which originated primarily from Pomraning and Titov, are additional incontestable evidence that the Markovian approach provides a convenient and appealing means for more impressive future accomplishments. The latter may include application of the Markovian approach to polarized radiation transport in a random mixture [49] of several (more than two) [50] immiscible materials.

ACKNOWLEDGMENTS

This manuscript has been authored by Battelle Memorial Institute, Pacific Northwest Division, under Contract No. DE-AC05-76RL01830 with the U.S. Department of Energy (DOE). The United States Government retains and the publisher, by accepting the article for publication, acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this manuscript, or allow others to do so, for United States Government purposes.

We dedicate this paper to the memory of two distinguished scientists, Gerald Pomraning and Georgii Titov, for having sparked our interest in the exciting field of stochastic RT and for their personal and scientific generosity.

REFERENCES

1. G. Basharin, A. Langville, and V. Naumov, “The life and work of A.A. Markov,” *Linear Algebra Appl*., **386**, pp.3-26 (2004).
2. W. Kendall, F. Liang, and J.-S. Wang (Eds.), *Markov Chain Monte Carlo: Innovations and Applications*, World Scientific Publishing (2005).
3. R. Farnoosh, and M. Ebrahimi, “Monte Carlo method for solving Fredholm integral equations of second kind,” *Appl. Math. Computation*, **195**, pp.309-315 (2008).
4. C. Chandrasekhar, “Stochastic Problems in Physics and Astronomy,” *Reviews of Modern Physics*, **15**, pp.2-89 (1943).
5. S. Brush, “History of the Lenz-Ising model,” *Reviews of Modern Physics*, **39**, pp.883-895 (1967).
6. J. Moussouris, “Gibbs and Markov systems with constrains,” *Journal of Statistical Physics*, **10**, pp.11-33 (1974).
7. G. Pomraning, *Linear Kinetic Theory and Particle Transport in Stochastic Mixtures*, World Scientific Publishing (1991).
8. G. Pomraning, “Cloud-radiation interactions: The Titov and other models,” *J. Atmos. Ocean. Opt*., **12**, pp.204-210 (1999).
9. G. Titov, “Statistical description of radiation transfer in clouds,” *J Atmos. Sci*., **47**, pp.24-38 (1990).
10. F. Malvagi, R. Byrne, G. Pomraning, R. C. J. Somerville, “Stochastic radiative transfer in a partially cloudy atmosphere,” *J. Atmos. Sci*., **50,** pp.2146-2158 (1993).
11. K.N. Liou, *An Introduction to Atmospheric Radiation*, Academic Press, San Diego (2002).
12. A. Marshak, and A.B. Davis, *3D Radiative Transfer in Cloudy Atmospheres*, Springer, New York (2005).
13. A. Kokhanovsky**,** and G. De Leeuw**,** *Satellite Aerosol Remote Sensing Over Land*, Praxis Publishing, UK (2009).
14. M.I. Mishchenko, “Multiple scattering, radiative transfer, and weak localization in discrete random media: Unified microphysical approach,” *Rev. Geophys.*, **46**, RG2003, doi:10.1029/2007RG000230 (2008).
15. K.F. Evans, “A general solution for stochastic radiative transfer,” *Geophys. Res. Lett*., **20**, pp.2075-2078 (1993).

16. B. Cairns, A.A. Lacis, and B.E. Carlson, ”Absorption within inhomogeneous clouds and its parameterization in general circulation models,” *J. Atmos. Sci*., **57**, pp.700–714 (2000).
17. V. Venema, F. Ament, and C. Simmer, “A Stochastic Iterative Amplitude Adjusted Fourier Transform Algorithm with improved accuracy,” *Nonlinear Processes in Geophysics,* **13**, pp.247-363 (2006).
18. K.F. Evans, and W.J. Wiscombe, “An algorithm for generating stochastic cloud fieldsfrom radar profile statistics,” *Atmos. Res.* **72,** 263–289 (2004).
19. P. Raisanen, H. W. Barker, M. Khairoutdinov, and D. A. Randall, “Stochastic generation of subgrid-scale cloudy columns for large-scale models,” *Q. J. R. Meteorol. Soc.*, **130,** pp.2047-2068 (2004).
20. V.A. Ogorodnikov and S.M.Prigarin, *Numerical Modelling of Random Processes and Fields: Algorithms and Applications*. VSP, Utrecht, the Netherlands (1996).
21. E.I. Kassianov, C. Long, and M. Ovtchinnikov, “Cloud sky cover versus cloud fraction: whole-sky simulations and observations,” *J. Appl. Meteor.*, **44**, pp.86-89 (2005).
22. P. Gabriel, and K.F. Evans, “Simple radiative transfer methods for calculating domain-averaged solar fluxes in inhomogeneous clouds,” *J. Atmos. Sci*., **53**, pp.858-877 (1996).
23. D. Xiu and G.E. Karniadakis, “The Wiener-Askey polynomial chaos for stochastic differential equations,” *SIAM J. Sci. Comput*., **24**, pp.619-644 (2002).
24. G.L. Stephens, P. Gabriel, S.C. Tsay, “Statistical radiaive transport in one-dimensional media and its application to the terrestrial atmosphere,” *J. Transport Th. Statist. Phys*., **20,** pp.139-175 (1991).
25. A. Borovoi**,** “On the extinction of radiation by a homogeneous but spatially correlated random medium**:** comment,” *J. Opt. Soc. Am. A*, **19**, pp.2517-2520 (2002).
26. A. B. Davis and A. Marshak, “Photon propagation in heterogeneous optical media with spatial correlations: enhanced mean-free-paths and wider-than-exponential free-path distributions,” *J. Quant. Spectrosc. Radiat. Transfer,* **84**, pp.3-34 (2004).
27. O. Anisimov and L. Fukshansky, “Stochastic radiation in macroheterogeneous random optical media,” *J. Quant. Spectrosc. Radiat. Transfer*, **48**, pp.169-186 (1992).
28. G.L. Stephens, “Radiative Transfer through Arbitrarily Shaped Optical Media. Part II. Group Theory and Simple Closures,” *J. Atmos. Sci*., **45**, pp.1837–1849 (1988).
29. M.M.R. Williams, *Random Processes in Nuclear Reactors*, Pergamon Press, New York (1974).
30. J.S. Cassell and M.M.R. Williams, “An approximate method for solving radiation and neutron transport problems in spatially stochastic media,” *Annals of Nuclear Energy*, **35**, 790-803 (2008).
31. S. A. El-Wakil, A. R. Degheidy and M. Sallah, “Time-dependent radiation transfer in a semi-infinite stochastic medium with Rayleigh scattering,” *J. Quant. Spectrosc. Radiat. Transfer,* **85**, pp.13-24 (2004).
32. M. J. Foster, and D. E. Veron, “Evaluating the stochastic approach to shortwave radiative transfer in the tropical western Pacific,” *J. Geophys. Res.*, 113, D22205, doi:10.1029/2007JD009581 (2008).
33. O.A. Avaste, and G.M. Vainikko, “Solar radiation transfer in broken clouds,” *Izv Acad Sci USSR Atmos. Oceanic. Phys*., **10**, pp.1054-1061 (1974).
34. V.E. Zuev, and G. Titov, “Radiative transfer in cloud fields with random geometry,” *J. Atmos. Sci*., **52,** pp.176-190 (1995).

35. D. Lane-Veron, and R. C. J. Somerville, "Stochastic theory of radiative transfer through generalized cloud fields," *J. Geophys. Res.*, **109**, D18113, doi:10.1029/2004JD004524 (2004).
36. D. Huang, Y. Knyazikhin, W. Wang, D. W. Deering, P. Stenberg, N. Shabanov, Bin Tan, and R. B. Myneni, "Stochastic transport theory for investigating the three-dimensional canopy structure from space measurements," *Remote Sensing of Environment*, **112**, pp.35-50 (2008).
37. E.I. Kasyanov, G.A. Titov, "Influence of scattering on long-wave radiation transfer through broken clouds," *Atmos. Opt*., **2**, pp.102-108 (1989).
38. V.E. Zuev, G.M. Krekov, M.M. Krekova, and G.A. Titov, "Mean characteristics of lidar signals from broken clouds," *J. Appl. Opt*., **26**, pp.3018-3025 (1987).
39. L.A.T., Machado, and W. B. Rossow, "Structural characteristics of radiative properties and tropical cloud clusters," *Mon. Wea. Rev*., **121**, pp.3234-3260 (1993).
40. R.F. Cahalan, and J. H. Joseph, "Fractal statistics of cloud fields," *Mon. Wea. Rev*., **117**, pp.261-272 (1989).
41. V.G. Plank, "The size distribution of cumulus clouds in representative Florida populations," *J. Appl. Meteor*., **8**, pp.46-67 (1969).
42. D.E. Lane, K. Goris, R. C. J. Somerville, "Radiative transfer through broken clouds: Observations and model validation," *J. Climate*, **15**, pp.2921-2933 (2002).
43. L.K. Berg, and E.I. Kassianov, "Temporal Variability of Fair-Weather Cumulus Statistics at the ACRF SGP Site," *J. Climate*, **21**, pp.3344–3358 (2008).
44. R.J. Hogan, and A.J. Illingworth, "Deriving cloud overlap statistics from radar," *Q.J.R. Meteorol. Soc*., **126**, pp.2903-2909 (2000).
45. L. Oreopoulos, and M. Khairoutdinov, "Overlap properties of clouds generated by a cloud-resolving model," *J. Geophys. Res*., **108**, doi:10.1029/2002JD003329 (2003).
46. E.I. Kassianov, "Stochastic radiative transfer in multilayer broken clouds. Part I.: Markovian approach," *J. Quant. Spectrosc. Radiat. Transfer*, **77**, pp.373-393 (2003).
47. E.I. Kassianov, T. Ackerman, R. Marchand, and M. Ovtchinnikov, "Stochastic radiative transfer in multilayer broken clouds. Part II.: Validation tests," *J. Quant. Spectrosc. Radiat. Transfer*, **77**, pp.395-416 (2003).
48. E.I. Kassianov, T. Ackerman, and P. Kollias, "The role of cloud-scale resolution on radiative properties of oceanic cumulus clouds," *J. Quant. Spectrosc. Radiat. Transfer*, **91**, pp.211-226 (2005).
49. M. Sallah, "Polarized radiation transfer in finite atmospheric media," *Planetary and Space Science*, **55**, pp.1283-1289 (2007).
50. M.M.R. Williams, "A stochastic *ansatz* and its relationship with the dichotomic Markov process," *Physica A: Statistical Mechanics and its Applications*, **387**, pp.4997-5002 (2008).