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TITLE OF PAPER: RADIATION TRANSPORT THROUGH RANDOM MEDIA

Guillaume Bal and Wenjia Jing
Department of Applied Physics and Applied Mathematics
Columbia University
500 W. 120th St, New York, NY 10027
gb2030@columbia.edu; wj2136@columbia.edu

ABSTRACT

We report on recent results obtained in the analysis of solutions of radiative transfer equations with random optical coefficients. We look at correctors beyond the mean field (homogenization) approximation and obtain such correctors in some cases as an application of the central limit theorem with correlated random variables. Applications of the theory include radiation through the atmosphere, neutron propagation in pebble bed nuclear and photon propagation in human and animal tissues as they appear in medical imaging.

Key Words: radiative transfer, random media, homogenization, random correctors, central limit result

1. INTRODUCTION

Radiative transfer equations are well-established in many fields of wave and particle propagation through heterogeneous media. What may be less studied and understood is propagation in media with high spatial variability that may not be known explicitly. In such cases, it is often appropriate to model such a high variability as random. We are then faced with solving radiative transfer equations with random optical parameters (random absorption and random scattering coefficients). We are interested in cases where these parameters oscillate at a much smaller spatial scale than the overall dimension of the domain where propagation is observed. We may thus introduce a small parameter ε equal to the correlation length of the randomness in the medium divided by the overall dimension of the domain. We are interested in the limit where ε is sent to zero.

The homogenization theory for radiative transfer equations with random coefficients is well-established. We refer the reader to e.g. [5–7] for a mean field (homogenization) theory in the transport and diffusive regimes of particle propagation. What is less understood is the random fluctuations that arise about the homogenized solution.

We consider here the situation where the absorption and scattering coefficients are random, highly oscillatory, stationary processes. Their correlation length is of order $\varepsilon \ll 1$ and we want to analyze the limiting framework where the latter small parameter tends to zero. We consider a regime of not-too-strong scattering so that propagation does not occur in a diffusive environment. At the leading order, we recover the homogenization results obtained in [6]. The homogenization theory is relatively straightforward: the effective absorption and scattering coefficients are nothing but the ensemble average (mathematical expectation) of the corresponding heterogeneous processes. The transport equations we have considered take the form

$$\theta \cdot \nabla_x u + \sigma(\frac{x}{\varepsilon}; \omega) u = k(\frac{x}{\varepsilon}; \omega) \int_{S^{d-1}} u(x, \theta') d\theta', \qquad (1)$$

with appropriate incoming boundary conditions at the domain's boundary ∂X, where $x \in X \subset \mathbb{R}^d$ with d the spatial dimension and $\theta \in S^{d-1}$ the direction of the particles. Here $\sigma(x;\omega)$ is the absorption coefficient at position $x \in X$ for realization $\omega \in \Omega$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, while $k(x;\omega)$ is the scattering coefficient, which we assume here is isotropic to simplify.

The developed theory may find important applications in several areas. A natural field of application is radiation through cloudy atmosphere when the density of clouds is neither negligible nor too large so that transport may be seen as a mixture of domains with thin and thick optical depths. A second potential application pertains to nuclear reactor physics and the new generation of pebble bed reactors, where neutrons undergo significant scattering inside the pebbles and essentially no scattering outside of the pebbles. Significant modifications to the theory would be necessary because the correlation length in the random medium (the size of the pebbles) is significantly larger than the mean free path (the mean distance between successive interactions of the neutrons with the underlying medium). A third potential application is in medical imaging and the analysis of photon propagation in human or animal tissues. The variations of the optical parameters at scales below the resolution limit nonetheless generate random fluctuations that our theory models asymptotically.

2. ASYMPTOTIC RESULTS

The transport solution may then be appropriately written as a Neumann expansion in the small parameter ε. The methodology roughly follows from that in [1], where elliptic problems are considered. Several terms then arise, which give rise to lower-order deterministic and random corrections. The correctors are explicitly described in several ”observation” or ”measurement” configurations.

Typical results presented in detail in [3] and involving somewhat lengthy formulas that we do not present here, are as follows. Measurements performed at one point in space for one direction of propagation are shown to have a standard deviation proportional to $\sqrt{\varepsilon}$. The reason may be seen as an application of the central limit theorem with correlated random variables. While homogenization may be seen as an application of the law of large numbers, correctors may be seen as the random fluctuation about that mean. For appropriately decorrelating random processes σ and k, the central limit result applies and gives rise to correctors of order $\sqrt{\varepsilon}$ after integration along a line (the path of the particles).

Measurements averaged over space and angles have a significantly smaller standard deviation equal to $\varepsilon^{\frac{d}{2}}$, where d is spatial dimension (in practice, we consider dimensions $d = 2$ or $d = 3$). The reason is that the measurement involves particles that have traveled along a continuum of trajectories. The central limit then applies and it may be seen that the measurements are the sum of a number proportional to ε^{-d} of random variables, whence the above scaling.

This result shows that the randomness in the measurements depends on what one is interested in measuring. Spatially and angularly averaged measurements are significantly more stable than point-wise measurements in the space of positions and directions. Intermediate results apply to e.g. spatially point-wise angularly averaged measurements.

The asymptotic results also show how various measurements may be correlated. This has interesting applications in inverse problems, where the optical parameters in a domain are sought from available measurements. Typically, the random fluctuations oscillate at too small a scale to be reconstructed from inevitably noisy data. However, they generate noise in the data, and having an approximate understanding of the correlation of that noise helps devise minimum variance inversion algorithms that perform significantly better than when such correlations are not modeled at all. We refer the interested reader to e.g. [4] for an application of such ideas in the context of inverse spectral problems.

Homogenization theory is valid in transport under very mild assumptions on the optical parameters. They essentially need to be stationary and ergodic in order for the theory to apply; see [6]. The results we obtain for the random fluctuations need significantly stronger assumptions on the random coefficients. In the case of elliptic equations (for instance in the diffusive regime with random absorption and diffusion coefficients), it is shown in [1] that appropriate strongly mixing properties of the random coefficients are sufficient. Such assumptions are not too restrictive and essentially correspond to stipulating that the random fluctuations have an integrable correlation function. When the coefficients do not have integrable correlation function, we know that the central limit results do not apply and other theories need be developed; see e.g. [2]. In the case of transport equations, strong mixing assumptions are not sufficient for us to obtain our results. We need additional, stronger, assumptions on the structure of the fourth-order moments of the random coefficients. Such assumptions have been verified for several classes of processes, including Poisson point processes and processes that may be written as deterministic functionals of Gaussian processes. Simple mixtures of states are then included in the theory we have obtained.

As a final comment, we also obtain in the asymptotic analysis deterministic correctors of order $O(\varepsilon)$ beyond the homogenized solutions. Such correctors are therefore of comparable order to the random fluctuations in dimension $d = 2$ and are larger than the random fluctuations in dimension $d = 3$. We have explicit, albeit lengthy, expressions for such correctors, which strongly depend on fourth-order correlations of the random coefficients σ and k; see [3] for the details.

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