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RADIATION TRANSPORT THROUGH RANDOM MEDIA REPRESENTED AS MEASURABLE FUNCTIONS: POSITIVE VERSUS NEGATIVE SPATIAL CORRELATIONS

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ABSTRACT

We investigate particle transport through media with randomly variable material properties with a special emphasis on very high spatial frequencies. In multi-dimensional transport theory, particularly in computational studies, collision coefficients are assumed to be "functions" in the conventional sense of the word: they have specific values at every point in the medium. This assumption actually constrains spatial variability in the high-frequency range in a manner that is not a mathematical necessity and relaxing it opens the door to interesting transport physics. From the standpoint of the integral transport equation, coefficient variations in space can indeed be represented as "measures," i.e., mathematical constructs that only make *numerical* sense under an integral (in the sense of Lebesgue). The new results are based on a variability model consisting of a zero-mean Gaussian scaling noise riding on a constant value large enough with respect to the amplitude of the noise to yield overwhelmingly non-negative coefficients; technically, this represents a "measurable" function. We first generalize known results about sub-exponential transmission from regular (almost everywhere continuous) to merely measurable functions with positively-correlated fluctuations. Another interesting outcome is that, in this very general measure-based framework, one can use conventional (continuum-limit) transport theory to address negatively- as well as positively-correlated stochastic media. We thus resolve a controversy concerning recent claims that only discrete-point process approaches can accommodate negative correlations, i.e., anti-clustering of the obstructing material particles. We obtain in this case the predicted super-exponential behavior, but only over a limited range of transport distances.

Key Words: multi-dimensional radiation transport, correlated stochastic media, non-exponential transmission, high spatial frequencies, clustering/anti-clustering material

1. INTRODUCTION AND OVERVIEW

Transport media whose material properties can only be described in practice by statistical methods are playing increasingly important roles in nuclear engineering (e.g., next generation pebble-bed reactors), in medical physics (e.g., dosimetry and tomography), in atmospheric radiative transfer (e.g., the role of clouds in large-scale fluxes), and so on. Two broad classes of solutions have arisen for such transport problems: homogenization and, broadly speaking,

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alternate transport theories. In the former pursuit, one seeks ways of redefining the material properties of the medium, as if it were uniform at the (usually large) scales of interest, but in a manner that accounts for the relevant effects of smaller-scale (usually unresolved) variability. In the later approach, one arrives at new transport equations to solve analytically or numerically. Homogenization (a.k.a. the “effective medium” approach) is very attractive because it reduces the difficult multi-dimensional problem to a much simpler problem for a uniform medium, which has known solutions (often in 1D, using slab geometry). Although the pose new technical challenges, new transport equations describing the stochastic transport problem are generally a more realistic/accurate approach, a well-known example being the problem of Markovian binary mixtures [1].

An important preliminary problem is the characterization of elementary transport processes such as radiation propagation in heterogeneous media between emission, scattering, absorption, and detection/escape events. Studies are on-going, for instance, in chord-length distributions for media made of closely packed disks or spheres [2]. A recent series of papers has provoked a controversy about non-exponential transmission laws, which impact directly all transport processes, largely because of the unconventional description of the propagation part of transport problem in terms of “discrete-point process” theory rather than the traditional “continuum” theory (encapsulated in the linear Boltzmann equation). Introducing discrete-point process modeling into transport through heterogeneous media, Kostinski [3] argued strongly that spatial correlations will lead to sub-exponential behavior in the law of direct transmission (a.k.a. free-path distribution), which is known to be exponential in uniform media (Beer’s law). His findings were critiqued by Borovoi [4] who relied on classic continuum theory. In his reply, Kostinski [5] insists that the discrete-point approach is more fundamental and, to make his case, he claims that the case of transport in negatively-correlated (a.k.a “super-homogeneous”) media can be modeled by discrete-point methods and not by continuum methods, citing related work by Shaw et al. [6]. One of the present authors weighed in very strongly favoring non-exponential mean transmission laws for correlated media [7], using conventional (continuum-based) radiation transport theory. This leaves the issue of negatively-correlated media open to debate—one that we resolve in the course of the present investigation (cf. Section 5.2).

In the meantime, non-exponential transmission laws are finding they way into specific applications. For instance, Davis [8, and references therein] constructs a new transport equation, with an *anomalous* diffusion limit, that explains recent ground-based observations of time-domain transport in the Earth’s cloudy atmosphere. As another instance, Larsen [9] derives by way of a careful asymptotic analysis a homogenized (but otherwise standard) diffusion theory that accounts for deviations of the mean transmission law from the exponential case.

In the following section, we pose the general problem of radiation transport in heterogeneous 3D media from the standpoint of the integral transport equation, highlighting the key role of the transmission factor (i.e., propagation part of the linear transport kernel). In Section 3, we survey previous results on the non-exponential transmission laws that follow directly from statistical analysis of the propagation kernel. In Section 4, the spatial variability model we adopt is presented and its general properties are described. In Section 5, we derive the transmission laws for previously unexplored classes of media in the model space—those where the largest fluctuation amplitudes are at the highest frequencies—and we discuss some ramifications. We draw our conclusions in Section 6.

2. 3D RADIATION TRANSPORT: THE INTEGRAL EQUATION PERSPECTIVE

Let $I(\vec{x},\vec{\Omega})$ denote steady-state radiance at position $\vec{x}$ in 3D space propagating into direction $\vec{\Omega}$; its physical units are W/m^2/sr. In monochromatic 3D radiative transfer (or one-group neutron transport), we seek to determine I in a convex region M $\subseteq \mathbb{R}^3$ (boundary ∂M) where material properties are defined by (*i*) the extinction coefficient $\sigma(\vec{x})$ for either scattering or absorption (or, in the case of neutrons, multiplication) and (*ii*) differential scattering cross-section (per unit of volume) $\mathrm{d}\sigma_\mathrm{s}/\mathrm{d}\vec{\Omega}(\vec{x};\vec{\Omega}' \to \vec{\Omega})$. Moreover, we are given the distributions of volume sources ($q_\mathrm{v}(\vec{x},\vec{\Omega})$, $\vec{x} \in$ M) and boundary sources ($q_\mathrm{b}(\vec{x},\vec{\Omega})$, $\vec{x} \in \partial$M, $\vec{\Omega}\cdot\hat{\mathbf{n}}(\vec{x}) < 0$ where $\hat{\mathbf{n}}(\vec{x})$ is the outward normal to ∂M at $\vec{x}$).

Particularly with numerical solutions in mind, a convenient way of determining the linear transport problem at hand is to use the integral equation

$$\begin{aligned} I(\vec{x},\vec{\Omega}) &= \int_0^{s_\mathrm{b}(\vec{x},-\vec{\Omega})} \mathrm{e}^{-\int_0^{s'} \sigma(\vec{x}-\vec{\Omega}s'')\mathrm{d}s''} \left[\int_{4\pi} \frac{\mathrm{d}\sigma_\mathrm{s}}{\mathrm{d}\vec{\Omega}}(\vec{x}-\vec{\Omega}s';\vec{\Omega}'\to\vec{\Omega})\, I(\vec{x}-\vec{\Omega}s',\vec{\Omega}')\,\mathrm{d}\vec{\Omega}' \right] \mathrm{d}s' \\ &+ Q(\vec{x},\vec{\Omega}) \end{aligned} \tag{1}$$

where $s_\mathrm{b}(\vec{x},-\vec{\Omega})$ is the distance along the upwind beam $\{\vec{x},-\vec{\Omega}\}$ from $\vec{x}$ to its unique intersection with ∂M. The source term denoted by $Q(\vec{x},\vec{\Omega})$ in the above can be computed from given volume- and boundary-source distributions:

$$Q(\vec{x},\vec{\Omega}) = \int_0^{s_\mathrm{b}(\vec{x},-\vec{\Omega})} q_\mathrm{v}(\vec{x}-\vec{\Omega}s')\mathrm{e}^{-\int_0^{s'}\sigma(\vec{x}-\vec{\Omega}s)\mathrm{d}s}\mathrm{d}s' \;+\; q_\mathrm{b}(\vec{x}-\vec{\Omega}s_\mathrm{b}(\vec{x},-\vec{\Omega}))\mathrm{e}^{-\int_0^{s_\mathrm{b}(\vec{x},-\vec{\Omega})}\sigma(\vec{x}-\vec{\Omega}s)\mathrm{d}s}. \tag{2}$$

As an example, $q_\mathrm{v}(\vec{x},\vec{\Omega})$ in atmospheric radiation transport would be used to model thermal sources inside the medium while $q_\mathrm{b}(\vec{x},\vec{\Omega})$ would be used to specify *isotropic* thermal emission by the underlying surface and the *unidirectional* solar irradiation at the top of the medium. In the latter case, one can usefully limit $I(\vec{x},\vec{\Omega})$ to once or multiply scattered radiation, for which boundary sources vanish, and use

$$q_\mathrm{v}(\vec{x},\vec{\Omega}) = F_0\,\frac{\mathrm{d}\sigma_\mathrm{s}}{\mathrm{d}\vec{\Omega}}(\vec{x};\vec{\Omega}_0\to\vec{\Omega})\,\mathrm{e}^{-\int_0^{s_\mathrm{b}(\vec{x},-\vec{\Omega}_0)}\sigma(\vec{x}-\vec{\Omega}_0 s)\mathrm{d}s}, \tag{3}$$

where F_0 is the solar constant (in W/m^2/μm) and $\vec{\Omega}_0$ is the direction of incidence of the solar beam.*

The important fact about (1)–(3) is the recurring appearance of

$$T(\vec{x}_0,\vec{\Omega};s) = \mathrm{e}^{-\int_0^s \sigma(\vec{x}_0+\vec{\Omega}s')\mathrm{d}s'}, \tag{4}$$

*Assuming slab geometry for simplicity, M = $\{\vec{x}\in\mathbb{R}^3; 0<z<h\}$ where slab thickness is denoted by h, (3) results from setting $q_\mathrm{b}(\vec{x},\vec{\Omega}) = F_0\delta(z)\delta(\vec{\Omega}_0-\vec{\Omega})$ (and $q_\mathrm{v}\equiv 0$) and once iterating (1)–(2), starting with $I_0 \equiv q_\mathrm{b}$.

which is the *local* law of direct transmission from $\vec{x}_0$ to $\vec{x}_1 = \vec{x}_0 + \vec{\Omega}s$, i.e., over distance $s = |\vec{x}_1 - \vec{x}_0|$ along $\vec{\Omega} = (\vec{x}_1 - \vec{x}_0)/|\vec{x}_1 - \vec{x}_0|$. This is the cumulative probability for the random distance from $\vec{x}_0$ to the next scattering or absorption event at $\vec{x}_1$ to exceed s. In stochastic media, $T(\vec{x}_0, \vec{\Omega}; s)$ or, equivalently, $T(\vec{x}_0, \vec{x}_1)$ is a critically important non-local quantity, which is itself a random number. We are keenly interested in the statistical properties of $T(\vec{x}_0, \vec{x}_1)$. This is the crucial propagation part of the linear transport kernel in (1) that remains even when the scattering is everywhere isotropic ($\mathrm{d}\sigma_\mathrm{s}/\mathrm{d}\vec{\Omega}(\vec{x}; \vec{\Omega}' \to \vec{\Omega}) \equiv \sigma_\mathrm{s}/4\pi$).

3. MEAN TRANSMISSION LAW: EXPONENTIAL OR NOT?

We will focus on the ensemble-average value of

$$T(\vec{x}_0, \vec{x}_1) = \exp\left[-\int_0^1 \sigma\left((1-u)\vec{x}_0 + u\vec{x}_1\right) \mathrm{d}\ell(u)\right] \tag{5}$$

which we denote $\langle T\rangle(s)$. This is a spatial statistic that will depend strongly on the nature of the correlations in the random optical medium. We anticipate that it will depend only on the transport distance $s = |\vec{x}_1 - \vec{x}_0|$, which will certainly be true in *statistically* homogeneous and isotropic media.

Davis and Marshak [7] investigated the properties of $\langle T\rangle(s)$ under very general assumptions about the variability of $\sigma(\vec{x})$. In essence, they only assumed that the natural notation "$\sigma(\vec{x})$" makes mathematical sense, in other words, that σ is a well-defined function of position $\vec{x}$. Now, elementary kinetic theory tells us that σ can be expressed as the density of particles in the media times their total (scattering+absorption) cross-section. Assuming the same cross-section for all the particles, the spatial variability of σ just reflects that of the material density. Density is a continuum notion that requires a specific connection between finite numbers of particles in varying volumes around $\vec{x}$. Specifically, we naively require that

$$\bar{\sigma}_r(\vec{x}) = \frac{1}{4\pi r^3/3} \iiint_{|\vec{x}'-\vec{x}|<r} \sigma(\vec{x}')\mathrm{d}\vec{x}' \tag{6}$$

does not depend on r, if small enough. This requirement is expected to hold at least to within the Poissonian fluctuations that are to be expected when the volume's dimensions become commensurate with the inter-particle distance. It is the same as to ask that

$$\overline{\sigma}(\vec{x}, \vec{\Omega}; r) = \frac{1}{r}\int_0^r \sigma(\vec{x} + \vec{\Omega}s')\mathrm{d}r' \equiv \bar{\sigma}_0(\vec{x}) \tag{7}$$

for any choice of $\vec{\Omega}$. Davis and Marshak called this presumably benign property of the medium "one-point scale independence."

Letting $\langle\cdots\rangle$ denote ensemble averages, Davis and Marshak derived three general results for 1-point scale-independent media:

- The mean direct transmission law $\langle T\rangle(s)$ is exponential only if $\sigma(\vec{x})$ is uniform in M. This is the non-trivial converse of the elementary result showing that, if σ is uniform, then $T(s) = \exp(-\sigma s)$.
- Recalling that the ensemble average free-path distribution is $p(s) = |\mathrm{d}\langle T\rangle/\mathrm{d}s|$, the mean-free-path (MFP) given by $\langle T\rangle(s)$, namely, $\ell = \mathcal{E}(s) = \int_0^\infty sp(s)\mathrm{d}s$, is always larger than the prediction from the exponential distribution based on the mean extinction, $1/\langle\sigma\rangle$.
- $\langle T\rangle(s)$ is always sub-exponential in the sense that, if one uses the above MFP ℓ to predict higher-order moments $\mathcal{E}(s^q) = \int_0^\infty s^q p(s)\mathrm{d}s$ for $q = 2, 3, \ldots$, then the exponential assumption yields an underestimate. In other words, $\mathcal{E}(s^q) \geq q!\ell$ for $q > 1$, where "=" applies only when the medium is uniform.

This last item is truly a statement about the tail of the free-path distribution, that is, how $\langle T\rangle(s)$ decays as $s \to \infty$. All of the above results follow from various applications of Jensen's inequality [10].

The present study is about relaxing the unnecessarily restrictive assumption of 1-point scale-independence. This generalization is possible, but the price is that we can no longer think of σ as a *regular* function (i.e., with continuity almost everywhere), which is numerically well-defined at every point. Since it only appears under line integrals in (1)–(2), we can assume σ is only a *measurable* function, or simply a measure on M.

4. ADOPTED VARIABILITY MODEL

Leaving a completely general approach for future work, we will concentrate here on a representative and relevant class of stochastic optical media where the extinction field is defined as a random measurable function denoted "$\sigma(x, \mathrm{d}x)$" that has two components: (1) a regular, indeed constant, part $\langle\sigma\rangle > 0$ and (2) a noise, which we take as Gaussian with zero mean and variance that scales as a power law with wavenumber in Fourier space. We recall that, for this type of function, only definite integrals of $\sigma(x, \mathrm{d}x)$ have a well-defined numerical value; in other words, we can only evaluate quantities such as

$$\tau(x_0, x_1) = \int_{x_0}^{x_1} \sigma(x, \mathrm{d}x) \tag{8}$$

that we interpret in the sense of Lebesgue. In the present context, we identify this random variable with *optical* distance $\int_{x_0}^{x_1} \sigma(x)\mathrm{d}x$, which is usually understood as a standard Riemann integral.

4.1. Formal Definition in Fourier Space

Our model is best defined in Fourier space where, because of (7), we can work in 1D without loss of generality. Letting wavenumber be denoted by $k \in \mathbb{R}$, we want the energy spectrum of σ to be

$$E_\sigma(k)\mathrm{d}k = \|\hat{\sigma}(k, \mathrm{d}k)\|^2 \sim k^{-\beta}\mathrm{d}k, \tag{9}$$

where the spectral exponent β can cover a significant range of values, both positive and negative.

We thus assume

$$\hat{\sigma}(k, \mathrm{d}k) = \mathrm{F}[\sigma(x, \mathrm{d}x)] = \int_{-\infty}^{+\infty} \mathrm{e}^{+\mathrm{i}kx}\sigma(x, \mathrm{d}x) = c\, k^{-\beta/2}\, g_k \mathrm{e}^{\mathrm{i}\phi_k} \sqrt{\mathrm{d}k} \tag{10}$$

for $k > 0$, and $\hat{\sigma}(0, \mathrm{d}k) = \langle\sigma\rangle \mathrm{d}k$; for $k < 0$, we take the complex conjugate of (10) to ensure a real-valued inverse Fourier transform. In the above, g_k denotes a zero-mean/unit-variance Gaussian random amplitude $N(0, 1)$ for the Fourier mode. We have introduced here the notation

$$N(\mu, \sqrt{\text{variance}})$$

for a Gaussian random variable with mean and standard deviation in the 1st and 2nd arguments, and ϕ_k is its uniform random phase in $[0, 2\pi)$. The exponent β defines the scaling property, with qualitative ramifications discussed in the next subsection, while c is an overall variability strength parameter that will be specified further on, bearing in mind the following constraint. In the present application, we will tune c as a function of $\langle\sigma\rangle$ and β to keep integrals of

$$\sigma(x, \mathrm{d}x) = \mathrm{F}^{-1}[\hat{\sigma}(k, \mathrm{d}k)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kx}\hat{\sigma}(k, \mathrm{d}k) \tag{11}$$

non-negative for a vast majority of possible positions x and integration domains.

4.2. Nomenclature

The key variability parameters in our model are (1) c for 1-point statistics of the extinction field such as the probability density function (PDF) and/or moments thereof, and (2) β that determines 2-point (spatial correlation) statistics. Qualitative differences occur as β is varied, from large to small values:

1. for $\beta \geq 3$, every realization of $\sigma(x)$ is a random but smooth (almost everywhere differentiable) function with, in particular, no discontinuities (otherwise β jumps to 2);
2. for $2 < \beta < 3$, σ is a fractional Brownian motion (fBm) with "persistence" in the sense that [11] $\langle[\sigma(x + 2r) - \sigma(x + r)][\sigma(x + r) - \sigma(x)]\rangle > 0$ for any two successive increments at any scale r;
3. for $\beta = 2$, σ is the spatial counterpart of classic Brownian motion (Bm, a.k.a. the Weiner-Lévy process) where successive increments at any scale are independent, i.e., $\langle[\sigma(x + 2r) - \sigma(x + r)][\sigma(x + r) - \sigma(x)]\rangle = 0$;
4. for $1 < \beta < 2$, σ is a fractional Brownian motion with "anti-persistence" in the sense that $\langle[\sigma(x + 2r) - \sigma(x + r)][\sigma(x + r) - \sigma(x)]\rangle < 0$ for two successive increments at scale r, a scenario of tremendous interest in turbulent media such as clouds [12];
5. for $\beta = 1$, we have the special case of spatial "$1/f$" (a.k.a. "red") noise, borrowing the traditional time-domain notation, that is at the boundary between the above random fields with diverging variance due to large-r/small-k behavior (the "infra-red catastrophe") and the following ones where the divergence is at small-r/large-k's (the "ultra-violet catastrophe");

6. for $0 < \beta < 1$, σ is a field of "pink" noise that generates persistent fBm by integration (in practice, a division by $\mathrm{i}k$ in Fourier space);

7. for $\beta = 0$, σ is a field of "white" noise that generates standard Bm by integration;

8. for $-1 < \beta < 0$, σ is a field of "blue" noise that generates anti-persistent fBm by integration.

Taking a stochastic modeling perspective, the $\beta = 1$ case is a critical threshold between statistically stationary [†] fields when β is smaller, and non-stationary ones for β larger than unity. [‡] The noted connection, via definite integrals, between the stationary Gaussian scaling noises and the non-stationary fBm processes proves crucial further on.

4.3. Important Statistical Properties

When $0 < \beta < 1$, the stationary σ-field is discontinuous (almost everywhere), so one should use the $\sigma(x, \mathrm{d}x)$ notation for measures. The Wiener-Khinchin theorem [§] for scaling stationary noises tells us that a wavenumber energy spectra with a power-law decay in this range of parameter β leads to a power-law auto-correlation function [¶]

$$\gamma(r) = \langle[\sigma(x+r) - \langle\sigma\rangle][\sigma(x) - \langle\sigma\rangle]\rangle \sim r^{-\alpha_\beta} \tag{12}$$

independently of x (stationary process), where $\alpha_\beta = 1 - \beta$. We also have $0 < \alpha_\beta < 1$. The limit $\beta \to 0$, hence $\alpha_\beta \to 1$, leads to the well-known case of white noise where the Wiener-Khinchin theorem gives us $\delta(r)$ on the r.-h. side of (12): white noise is indeed "δ-correlated." The limit $\beta \to 1$, hence $\alpha_\beta \to 0$, leads to a flat $\gamma(r)$ marking the entry into the realm of random functions with "long-range" memory, that is, any realization of $\sigma(x)$ with $\beta > 1$.

When $1 < \beta < 3$, $\sigma(x)$ is a valid notation since it is a regular function that is stochastically continuous. Indeed, the Wiener–Khinchin theorem for such *nonstationary* processes, but with stationary increments [13, 14, and others], tells us that a wavenumber energy spectra with a power-law decay in this parameter range leads to an increasing power-law for the 2nd-order structure function [‖]

$$\mathrm{SF}_2(r) = \langle[\sigma(x+r) - \sigma(x)]^2\rangle \sim r^{2H_\beta} \tag{13}$$

independently of x (stationary increments), where

$$H_\beta = \frac{\beta - 1}{2} \text{ (hence } \beta_H = 2H + 1) \tag{14}$$

is the so-called Hurst exponent. We have $0 < H_\beta < 1$. As anticipated above, we see that the limit $\beta \to 3$ (hence $H_\beta \to 1$) leads to smooth/differentiable behavior for $\sigma(x)$, i.e., $|\sigma(x+r) - \sigma(x)| \sim r$ at typical points. We also note that (13) can be used to show that

$$\langle[\sigma(x+2r) - \sigma(x+r)][\sigma(x+r) - \sigma(x)]\rangle = [\mathrm{SF}_2(2r) - 2\mathrm{SF}_2(r)]/2 \sim (2^{2H_\beta - 1} - 1) \times r^{2H_\beta}, \tag{15}$$

[†] One should say here statistically *homogeneous* for random functions since they unfolding in space rather than time, but we will carry on with this wide-spread abuse of the time-domain language.

[‡] Throughout this study, we think of stationarity in the "broad" sense, meaning based on 2nd-order moments and spatial statistics.

[§] For broad-sense stationary processes, energy spectrum $E(k)$ and auto-correlation function $\gamma(r)$ form a Fourier transform pair.

[¶] Since this is abusive notation in the case of scaling noises, one actually defines $\gamma(r)$ as the inverse Fourier transform of the spectral measure $E(k, \mathrm{d}k) \sim k^{-\beta}\mathrm{d}k$ in (9).

[‖] a.k.a. semi-variogram (as follows from $\mathrm{SF}_2(r) = 2[(\langle\sigma^2\rangle - \langle\sigma\rangle^2) - \gamma(r)]$ in the case of (broad-sense) stationary processes) which justifies the signs assigned in the above §4.2 to this correlation between two successive increments in $\sigma(x)$ at scale r.

Finally, the statistical property expressed in (13) has a *local* version:

$$|\sigma(x+r)-\sigma(x)| \sim r^{h(x)}, \tag{16}$$

where $h(x)$ is the Hölder (regularity) exponent [12]. From a multi-resolution analysis perspective, we see that (7) is in essence a local constraint on the projection of $\sigma(x)$ onto the so-called scaling function (local *coarsened* value at scale r). In the same spirit, (16) is a constraint on the projection of $\sigma(x)$ onto the so-called wavelet function (local *detail* at scale r). It is clear that, for 1-point scale-independence (a coarsening property) to prevail, we need the detailed behavior to be characterized by $h(x) > 0$.

5. MEAN TRANSMISSION LAWS FOR UNEXLORED MODEL SPACE

5.1. Media Represented by Pink and White Noises

We are first interested in computing the mean transmission law

$$\langle T(x_0,x_1)\rangle = \langle \exp[-\tau(x_0,x_1)]\rangle, \tag{17}$$

from (8), as a function of propagation distance $s = |x_1 - x_0|$ for the above model based on scaling noises, but only when $0 \leq \beta < 1$. Equivalently, we assume $1/2 \leq H_{\beta+2} = (\beta+1)/2 \leq 1$ in (14) for the associated Brownian motion ($\beta = 0$) or persistent fBm. This is an immediate extension of Davis and Marshak's 2004 paper [7] on 3D optical media with $\beta > 1$ in our present notations.**

More specifically, we evaluate

$$\langle \mathrm{e}^{-\tau}\rangle(s) = \int_0^\infty \mathrm{e}^{-\tau}\mathrm{Pr}\{\tau, \mathrm{d}t|s\}, \tag{18}$$

where $\mathrm{Pr}\{\tau, \mathrm{d}t|s\}$ is the probability law for the random variable τ in (8) and (17) for fixed s, viewed here as a given parameter. We now note that the definite integral $\tau(x_0, x_1)$ of scaling noise is simply an increment of the associated fBm, and we know what its distribution is. It is normally distributed with a mean value of $\langle\sigma\rangle s$ and variance given by $\mathrm{SF}_2(s)$ in (13). For specificity, we will write

$$\mathrm{SF}_2(s) = C^2_{\langle\sigma\rangle,H} \times (\langle\sigma\rangle s)^{2H} \tag{19}$$

where $2H$ is short for $2H_{\beta+2} = \beta + 1$ from (14) and the non-dimensional parameter $C_{\langle\sigma\rangle,H}$ is related to c, the tuned amplitude parameter in (10) for specific values of $\langle\sigma\rangle$ and H (or β). In summary, we have

$$\tau(x_0,x_1) \overset{\mathrm{d}}{=} N\left(\langle\sigma\rangle s\,,\, C_{\langle\sigma\rangle,H}(\langle\sigma\rangle s)^H\right), \tag{20}$$

meaning "equal in distribution." As anticipated, we see that this model of optical variablity is somewhat flawed because the support of a Gaussian is all of $\mathbb{R}$ while $\tau(x_0, x_1)$ is necessarily positive.

**These authors however already treated (using a finite inner scale) the $\beta = 0$ case of white-noise/δ-correlated optical media, but this was just to provide a counterexample to their required 1-point scale independence property.

We also know how to compute the characteristic function of a Gaussian random variable:

$$\langle \mathrm{e}^{+\mathrm{i}\xi N(\mu,\sqrt{v})} \rangle = \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-(N-\mu)^2/2v}}{\sqrt{2\pi v}} \exp(-\mathrm{i}\xi N)\,\mathrm{d}N = \exp\left(-\mathrm{i}\mu\xi - v\xi^2/2\right). \tag{21}$$

We use this as an approximation to obtain (18), the approximation being an extension of the integration limits from $[0, \infty]$ to $[-\infty, +\infty]$. As stated earlier, the impact of this extension to (physically spurious) negative values of τ is small if the assumed mean extinction $\langle\sigma\rangle > 0$ is large enough in view of the variability amplitude parameter $C_{\langle\sigma\rangle,H}$. At any rate, we can control this adverse impact of the simplifying assumption of Gaussian variability.

Setting $\xi = \pm\mathrm{i}$ in (21), with $\mu = \langle\sigma\rangle s$ and $v = (C\langle\sigma\rangle, H\mu)^{2H}$, we get

$$\langle T\rangle(s) = \langle \mathrm{e}^{-\tau}\rangle(s) \approx \exp\left(-\langle\sigma\rangle s + C^2_{\langle\sigma\rangle,H}(\langle\sigma\rangle s)^{2H}/2\right). \tag{22}$$

The special case of δ-correlated fluctuating media is retrieved for $H = 1/2$, hence

$$\langle T\rangle(s) \approx \exp\left(-[1 - C^2_{\langle\sigma\rangle,H}/2] \times \langle\sigma\rangle s\right). \tag{23}$$

In other words, as noted by Davis and Marshak [7], there is an effective ("homogenized") value of σ, smaller than the mean by a relative factor of $C^2_{\langle\sigma\rangle,1/2}/2$, that accounts for the raw variability in complete absence of spatial correlations. Otherwise, we find a positive correction term to $-\langle\sigma\rangle s$ for $\ln\langle T\rangle(s)$: $C^2_{\langle\sigma\rangle,H}(\langle\sigma\rangle s)^{2H}/2$, with $2H > 1$. This extends to pink-noise media Davis and Marshak's finding of sub-exponential transmission laws for what is denoted as "$\beta > 1$" cases in the present study.

We note that the "correction" term in s^{2H} will eventually grow larger in magnitude than the presumably dominant term, $-\langle\sigma\rangle s$, if $H > 1/2$; the transport distance to this turn-around in $\langle T\rangle(s)$ is

$$s^\star = \frac{1}{(C^2_{\langle\sigma\rangle,H} H)^{1/(2H-1)}\langle\sigma\rangle}. \tag{24}$$

In reality, we never expect the derivative of $\langle T\rangle(s)$ to vanish at finite s (let alone start increasing with s). This is clearly an artifact of the Gaussian assumption (persistent fluctuations of τ into negative values) that becomes manifest at large values of s. In practice, we would never apply the model to such high values of s and/or of $C_{\langle\sigma\rangle,H}$.

5.2. Media Represented by Blue Noises

The above computation of $\langle T\rangle(s)$ for $H \geq 1/2$ ($0 \geq \beta < 1$) carries over wholesale to $0 < H < 1/2$ ($-1 < \beta < 0$). The fact that we no longer have a straightforward Weiner-Khinchin connection between spectral and physical-space statistics is not troublesome. We simply apply (22) to blue-noise media where the high-frequency fluctuations not only lead to an ultra-violet catastrophe but contain increasingly more strength as k increases.

In this case, the "correction" term in s^{2H} exceeds in magnitude the presumably dominant term

$-\langle\sigma\rangle s$ at very small values of s, specifically, if

$$0 < s < s_1 = \frac{(C^2_{\langle\sigma\rangle,H}/2)^{1/(1-2H)}}{\langle\sigma\rangle}. \tag{25}$$

In reality, we of course never expect $\langle T\rangle(s)$ to exceed unity. This is just another artifact of the Gaussian assumption. In sharp contrast with the $H > 1/2$ case, the fluctuations of τ into negative values become manifest at the smallest values of s. This, in turn, is an interesting consequence of the negative spatial correlations in blue noise that translates to negative correlations noted previously in successive increments of the associated fBm, cf. (15). Undesirable artifacts of the Gaussian variability model can be mitigated in a numerical implementation of the present model, for any value of H, by taking $\tau(x) = \max\{0, \int_0^x \sigma(x, \mathrm{d}x)\}$ for every realization of the noises.

Figure 1 summarizes our findings in a log-lin plot of $\langle T\rangle(s)$ versus $\langle\sigma\rangle s$, ranging from 0 to 2. We illustrate cases where $C_{\langle\sigma\rangle,H} = 0$, yielding the standard (uniform medium) model, and where $C_{\langle\sigma\rangle,H} = 1/\sqrt{2}$. In the latter case, the Hurst exponent of the associated fBm is taken to be $H = 1/4, 1/2, 3/4$. We see the modified exponential behavior for $H = 1/2$, the clear sub-exponential trend for $H > 1/2$ (with $\langle T\rangle(s) > \mathrm{e}^{-\langle\sigma\rangle s}$ for all $s > 0$), and the super-exponential behavior when $H < 1/2$ (actually, another modified exponential for large enough values of $\langle\sigma\rangle s$).

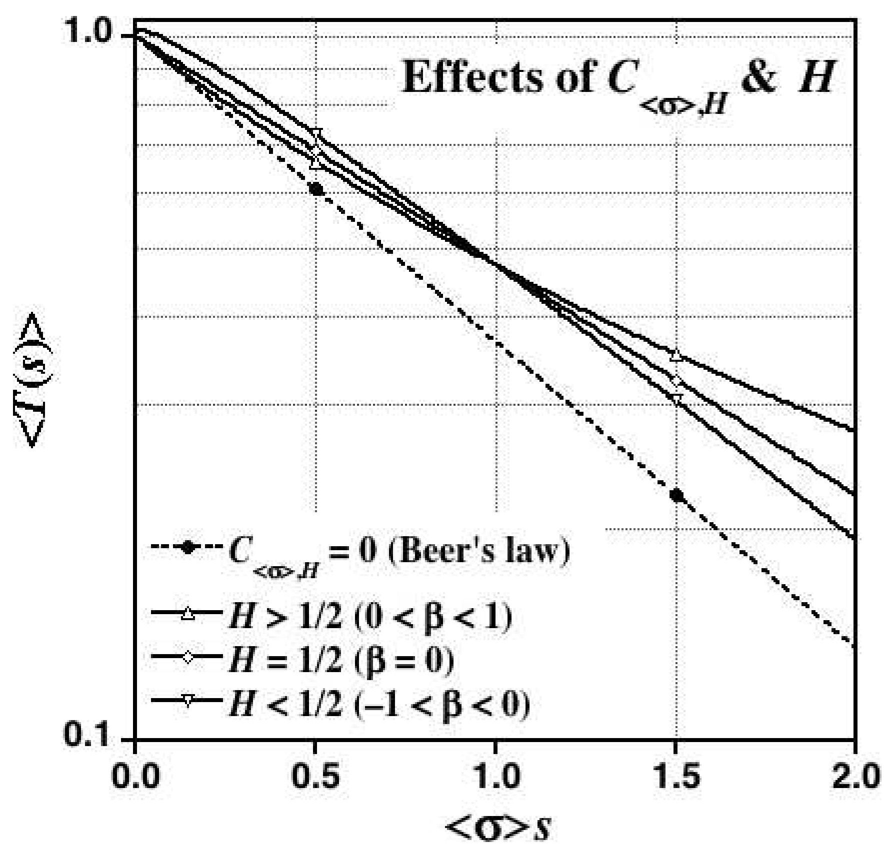

Figure 1. Mean transmission laws for stochastic media generated with white-, pink-, and blue-noises, compared to uniform media with the same mean extinction

In the present case, where $0 < H < 1/2$, the asymptotic (large s) behavior of $\langle T\rangle(s)$ is the standard prediction of Beer's law: $\exp(-\langle\sigma\rangle s)$. However, Shaw et al. [6] predict super-exponential behavior based on a discrete-point process analysis of scenarios where the material particles obstructing the flow of radiation are anti-clustered, i.e., they have negative (repelling) spatial correlations. We find this behavior, but only in the pre-asymptotic correction term, i.e., at propagation distances s less than a few times s_1 in (25), where it equals the dominant term.

However, one can also use the white-noise case in (23) as a benchmark to assess the effects of spatial correlations, in which case we see that the negatively-correlated noises lead for all practical purposes to exponential behavior, but modified from an extinction of $\langle\sigma\rangle[1 - C^2_{\langle\sigma\rangle,1/2}/2]$ to the *larger* value of $\langle\sigma\rangle$. The key Fig. 3 in Shaw et al.'s paper illustrates their numerical simulations of radiation propagation in various realizations of random media generated with a specific rule for obtaining negatively-correlated particle positions, at least at short distances. Rather than a super-exponential one per se, i.e., a qualitatively different decay rate (such as $\exp[-a(\langle\sigma\rangle s)^b]$ with $b > 1$), Shaw et al.'s plot shows what seems to be a modified exponential trend in $\langle T\rangle(s)$ with a steeper slope in log-lin axes than predicted by their version of Beer's law, presumably our white-noise case.

At any rate, this case-study establishes that negatively-correlated heterogeneous optical media can be handled with conventional radiative transfer, albeit at the cost of extending it from regular to merely measurable functions. This contradicts speculation by Kostinski, Shaw, and Lanterman, [3, 5, 6] that only discrete-point process approaches are general enough to accommodate such "super-homogeneous" media where distances between the material's particles are on average larger than predicted by a Poissonian distribution.

6. CONCLUSIONS

Using a specific-yet-representative case study of Gaussian scaling models, we have extended the general results published by Davis and Marshak [7] in 2004 about systematically sub-exponential transmission laws in heterogeneous optical media with (implicitly, positive) correlations. In that 2004 paper, fluctuations were represented by *regular* functions whereas here they are represented by *merely measurable* functions. In other words, we have gone from media where extinction has a well-defined numerical value at almost every point (a countable number of discontinuities can occur) to media where only integrals of the extinction field have well-defined numerical values (there can be discontinuities everywhere). In practice, we have gone from (at least stochastically) continuous media to very "noisy" media where high spatial frequencies dominate the variability.

Further extension from positively- to negatively-correlated fluctuating media (i.e., from decaying/pink to increasing/blue power-law wavenumber spectra) was also achieved, and we verified the prediction by Shaw et al. in 2002 [6] that this leads to super-exponential behavior for the mean transmission law, at least in comparison with the case of media with flat/white spectra. By the same token, we have established that, contrary to claims by Kostinski in 2002 [5], our approach grounded in the continuum framework of the linear Boltzmann equation can indeed accommodate negative correlations (anti-clustering of material particles) just as well as the discrete-point process approach that he originally introduced into transport theory in his 2001 paper [3] for positively-correlated media (material particle clustering).

We can now state with some confidence that there is complete equivalence between the classic continuum-based linear Boltzmann model for radiation transport in heterogeneous media and the interesting new description of transport fundamentals developed by Kostinski and coworkers at Michigan Technological University using discrete-point process theory. We are now eager to see how the recent derivation by Mishchenko [15, and several key references therein] of the multiple-scattering vector (polarization-capable) radiative transfer equation from the rigorous principles of statistical electromagnetic wave theory [††] can be extended to heterogeneous optical media with extinction fluctuations at all scales and amplitudes of interest. Indeed, Mishchenko's estimate of the impact on Beer's law of essentially sub-mean-free-path clumping is small. This is consistent with our prior prediction [7] and present theoretical results for 3D media dominated by high spatial frequency variability, i.e., white- and blue noises. It is also consistent with the "atomistic mixing" limit (vanishing correlation scale) in stochastic radiative transfer theory for Markovian binary media [1]. However, the interesting effects (severe perturbation of Beer's law) occur when the opacity fluctuation scales extend to the mean-free-path, which is itself boosted by heterogeneity, and beyond.

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[††] This new microphysical derivation is entirely classical and, as such, dismisses as highly misleading is the notion that one can substitute "photon" for, say, "neutron" in the definition of density/flux underlying the linear transport equation, and thus "derive" the radiative transfer equation from kinetic theory. From this strictly radiative transfer perspective, discrete-point process theory is even more at fault since it is all about transported particles interacting with material particles ... but this makes it all the more refreshing for particle transport per se.

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