LEAST SQUARES FINITE ELEMENTS ALGORITHMS IN THE SCEPTRE RADIATION TRANSPORT CODE

Clif Drumm and Wesley Fan
Sandia National Laboratories
Albuquerque, NM 87185-1179
crdrumm@sandia.gov; wcfan@sandia.gov

Andrew Bielen*
Department of Nuclear and Mechanical Engineering
The Pennsylvania State University
University Park, PA 20555-0001
andrew.bielen@nrc.gov

Jeffrey Chenhall
Nuclear Engineering and Radiological Sciences
The University of Michigan
Ann Arbor, MI 48109-2104
jchenhal@umich.edu

ABSTRACT

The least-squares finite-elements spherical-harmonics (Pn) and discrete-ordinates (Sn) solvers in the SCEPTRE radiation transport code are described. These solvers interface seamlessly with the first-order transport sweeps-base algorithm and the other second-order transport solvers (even-odd parity and self-adjoint angular flux) in SCEPTRE, using the same unstructured-mesh, angular quadrature and cross section framework, designed for massively-parallel computations. The least-squared methods are derived as a weighted-residual method, using the first-order transport operator to form weighting functions. The result is a symmetric positive definite linear system that can be solved efficiently with a parallel, preconditioned conjugate-gradients algorithm. Unlike the even-odd parity and self-adjoint angular flux implementations, the least-squares formulation does not break down in void or perfectly scattering regions. For the least-squares Pn algorithm, two alternative methods for imposing boundary conditions are described: one using a penalty method to minimize a residual on the boundary, and the other making use of the divergence theorem to transform volume integrals to terms involving boundary integrals. The moments integrals needed for the Pn algorithms are generated using recurrence relations, and the Lebedev angular quadrature is used to provide accurate half-range boundary integral terms on element surfaces up to order 130, needed for imposing the Pn boundary conditions. In particular, a Lebedev/Galerkin quadrature is developed using a Gram-Schmidt procedure for mapping between the discrete angular space and moments space, without loss of accuracy.

Key Words: Radiation transport, finite elements, least squares, spherical harmonics.

* Current address: U. S. Nuclear Regulatory Commission, Washington, DC 20555-0001
1. INTRODUCTION

Second-order transport methods, such as even-odd parity (EOP) [1] and self-adjoint angular flux (SAAF) [2], offer some advantages compared with first-order transport methods, e.g. the second-order transport operators are symmetric positive definite (SPD) and can be solved with the highly-efficient conjugate-gradients (CG) algorithm. Furthermore, the second-order transport algorithms implemented in the SCEPTRE code provide significant performance improvements over first-order methods for problems with extreme levels of scattering, such as electron transport[3], which is a main application for the SCEPTRE code. The least-squares (LS) finite-elements method (FEM) [4-6] offers an additional advantage in that the inverse of the removal/scattering operator is not needed, so that the method works for problems with void or pure scattering regions. Since many problems of interest for the SCEPTRE code involve charged-particle transport for problems with internal voids, this work provides a significant advance. Like SAAF, the LS algorithms make use of the full angular range, as opposed to EOP, which uses only half-range, so that the memory requirement is increased relative to EOP.

The SCEPTRE radiation transport code contains algorithms for sweeps-based first-order radiation transport and also LS, EOP and SAAF second-order transport, with either a discrete-ordinates ($S_n$) or spherical-harmonics ($P_n$) treatment of the angular dependence [7]. The different transport methods can be easily mixed and matched on a group-by-group basis, with the coupling provided by the down-scatter/fixed-source term. The methods share common finite-elements, angular-treatment, and cross-section tools, with the basic quantity of interest being the multi-group angular flux on a discontinuous finite-elements mesh. The second-order methods make use of the Trilinos software package[8], which enables both the discretization of the transport operator and the iterative solution of the resulting discretized operator to be performed with a high level of parallelism. For this reason, the various second-order methods implemented in SCEPTRE have comparable parallel performance.

Tools are provided to map back and forth between quantities based on discontinuous finite elements (used by the first-order transport algorithm) and continuous finite elements (used by the second-order transport algorithms). Tools are also provided to map back and forth between discrete-ordinates angular fluxes and angular flux moments (used by the $P_n$ solvers), using a Galerkin quadrature. The relative performance of the various algorithms in SCEPTRE is highly problem dependent, but the second-order methods generally perform better for charged-particle transport, as opposed to neutral-particle transport, and the LS solvers, the focus of this work, perform better for problems containing void or near void regions. In order to model a problem containing a void region, it is possible to use EOP or SAAF by approximating the void region with a low-density material, but this procedure significantly degrades the iterative convergence of EOP and SAAF, since the operator approaches singularity as the pseudo-void density approaches zero. For these types of problems, the LS algorithms greatly outperform EOP and SAAF.

The LS $P_n$ and $S_n$ algorithms in the SCEPTRE code are derived as a weighted-residual method, with weighting functions that are the angular flux basis functions operated on by the first-order transport operator. Imposing boundary conditions for the $S_n$ method is straightforward, but for $P_n$ the imposition is more complicated, since the flux moments are defined over an entire unit
sphere, while the boundary conditions are defined over only half of the unit sphere. Two methods
are described, one using a penalty method to minimize the residual for incoming directions on
the external boundaries, and the other using the divergence theorem to transform the volume
integral to terms involving boundary integrals. The two methods are described and compared.

The angular integrals needed for the \( P_n \) solvers are pre-computed based on recurrence relations
that have been developed for this work. The half-range boundary integrals needed for imposing
boundary conditions for the \( P_n \) solvers use a high-order Lebedev quadrature, which has been
implemented and tested in SCEPTRE up to order 130. A Lebedev/Galerkin quadrature set is
obtained using a Gram-Schmidt procedure to select a linearly-independent set of discrete
moments from among potential candidates.

Results are provided for a two-dimensional test problem containing an isolated source
surrounded by a void and reflector region [9]. Good agreement is obtained, comparing with
MCNP results, but the spatial convergence rates of the LS results are generally lower than
expected, which is not fully understood at the present time.

2. DERIVATION OF THE LEAST SQUARES APPROXIMATION

The steady-state, within-group Boltzmann transport equation for the angular flux \( \psi(r, \Omega) \) is

\[
\mathbf{\Omega} \cdot \nabla \psi + \mathcal{G} \psi(r, \Omega) = Q(r, \Omega), \text{ for } r \in V \text{ and } \Omega \in S^2
\]  

(1a)

where the removal/collision term is

\[
\mathcal{G} \psi = \sigma_c \psi - \int \sigma_x(r, \Omega' \cdot \mathbf{\Omega}) \psi(r, \Omega') d\Omega',
\]  

(1b)

and the fixed source \( Q(r, \Omega) \) contains any external and down-scatter (or up-scatter) sources. An
inflow boundary condition specifies the angular flux for all incoming directions on the physical
boundary \( \partial V \)

\[
\psi(r, \Omega) = \psi_b(r, \Omega), \text{ for } r \in \partial V \text{ and } \mathbf{n} \cdot \mathbf{\Omega} < 0.
\]  

(1c)

In order to solve the transport equation by a finite-elements approach, the angular flux is
expanded in a set of basis functions, \( u_n(r, \Omega) \), where the spatial dependence of the basis
functions is defined as a set of piece-wise linear or higher-order spatial basis functions that are
defined for individual finite elements. The angular dependence of the basis functions can be
defined as either a set of discrete directions in an \( S_n \) approximation, the spherical harmonics in a
\( P_n \) approximation, or some other angular approximation.

For a specified set of basis functions, the angular flux is approximated as

\[
\psi(r, \Omega) \approx \tilde{\psi}(r, \Omega) = \sum_{n'} \tilde{\psi}_{n'}(r, \Omega),
\]  

(2)
and the goal is to determine a set of coefficients, $\vec{\psi}$, to provide an "optimal" approximation to the angular flux. In a weighted-residual approach [4], the transport equation is multiplied by a set of weighting functions, $w_n(r, \Omega)$, and integrated over space and angle to generate a linear system that can be solved for the coefficients of the angular flux. For $P_n$, the angular dependence of the basis and weighting functions is described by the spherical harmonics. For a finite elements in angle treatment, the angular dependence would be described by a finite-elements discretization of the unit sphere. For $S_n$, the angular dependence is described by a generalized function that is similar to the Dirac $\delta$-function, with the additional restriction that the behavior of the square of the distribution is the same as the distribution itself.

To derive a LS approximation, the transport operator is applied to the angular-flux basis functions to generate a set of weighting functions, 

$$w_n(r, \Omega) = \Omega \cdot \nabla u_n(r, \Omega) + G u_n(r, \Omega).$$

Then, multiplying the transport equation by the weighting functions and integrating over space and angle results in a set equations,

$$\langle \Omega \cdot \nabla u_n + G u_n, \Omega \cdot \nabla \tilde{\psi} + G \tilde{\psi} \rangle = \langle \Omega \cdot \nabla u_n + G u_n, Q \rangle,$$

for all $n$  

(4)

or after substituting the expression for the approximate angular flux, Eq. (2),

$$\sum_{n'} \tilde{\psi}_{n'} \left[ \langle \Omega \cdot \nabla u_n, \Omega \cdot \nabla u_{n'} \rangle + \langle \Omega \cdot \nabla u_n, G u_{n'} \rangle + \langle G u_n, \Omega \cdot \nabla u_{n'} \rangle + \langle G u_n, G u_{n'} \rangle \right]$$

\(= \langle \Omega \cdot \nabla u_n, Q \rangle + \langle G u_n, Q \rangle \), for all $n$,

(5)

where $n$ is the number of unknown coefficients, which is equal to the number of spatial nodes in the mesh multiplied by the number of angular unknowns, either the number of discrete directions for $S_n$ or the number of angular moments for $P_n$.

### 2.1. Imposition of Boundary Conditions

For $S_n$ the boundary conditions, Eq. (1c), are imposed as Dirichlet boundary conditions, with minor modifications performed to maintain the symmetry of the system matrix. For a spherical-harmonics angular approximation the imposition of boundary conditions is more difficult, since the angular moments are defined over the full $4\pi$ angular range, while the boundary conditions are specified over only half of the angular range.

One possible approach for applying $P_n$ boundary conditions is to use the divergence theorem to convert the volume integrals to mixed volume-surface integrals, to allow the imposition of boundary conditions. Applying the divergence theorem to the first-order terms in Eq. (5) results in
\[
\langle \mathbf{\Omega} \cdot \nabla u_n, \mathbf{G} u_{n'} \rangle + \langle \mathbf{G} u_n, \mathbf{\Omega} \cdot \nabla u_{n'} \rangle \\
= -\langle u_n, \mathbf{\Omega} \cdot \nabla \mathbf{G} u_{n'} \rangle - \langle \mathbf{\Omega} \cdot \nabla \mathbf{G} u_n, u_{n'} \rangle + \int_L \int_{4\pi} \mathbf{\Omega} \cdot \mathbf{n} (u_n \mathbf{G} u_{n'} + \mathbf{G} u_n u_{n'}) \, d\Omega ds,
\]

(6)

where the surface integral is over the element surface \( L \), and \( \mathbf{n} \) is the unit outward normal on the element surface. For element surfaces on the external boundary \( \partial V \), the angular integration can be split in half: for incoming directions use the known values, \( \psi_b(r, \mathbf{\Omega}) \), of the angular flux, and for outgoing directions use the approximate angular flux, \( \tilde{\psi}(r, \mathbf{\Omega}) \).

Substituting Eqs. (6) into Eq. (5), after splitting the boundary surface integral into incoming and outgoing directions, results in the linear system,

\[
\sum_{n'} \tilde{\psi}_{n'} \left[ \langle \mathbf{\Omega} \cdot \nabla u_n, \mathbf{\Omega} \cdot \nabla u_{n'} \rangle - \langle u_n, \mathbf{\Omega} \cdot \nabla \mathbf{G} u_{n'} \rangle - \langle \mathbf{\Omega} \cdot \nabla \mathbf{G} u_n, u_{n'} \rangle + \langle \mathbf{G} u_n, \mathbf{G} u_{n'} \rangle \right. \\
+ \int_{L_{int}} \int_{4\pi} \mathbf{\Omega} \cdot \mathbf{n} (u_n \mathbf{G} u_{n'} + \mathbf{G} u_n u_{n'}) \, d\Omega ds \\
+ \int_{\partial V} \int_{\Omega \cdot n > 0} \mathbf{\Omega} \cdot \mathbf{n} (u_n \mathbf{G} u_{n'} + \mathbf{G} u_n u_{n'}) \, d\Omega ds \\n\left. \right] \\
= \langle \mathbf{\Omega} \cdot \nabla u_n, Q \rangle + \langle \mathbf{G} u_n, Q \rangle - \oint_{\partial V} \int_{\Omega \cdot n < 0} \mathbf{\Omega} \cdot \mathbf{n} (u_n \mathbf{G} \psi_b + \mathbf{G} u_n \psi_b) \, d\Omega ds,
\]

(7)

where \( L_{int} \) includes internal surfaces only on material interfaces (for uniform material properties across an interface, the boundary integral contributions cancel out and may be neglected).

### 2.2. Penalty Method of Imposing Boundary Conditions

An alternative is to use a penalty method to include boundary terms. On the physical boundary, the angular flux should approximate the known boundary values,

\[
\tilde{\psi}(r, \mathbf{\Omega}) = \sum_{n'} \tilde{\psi}_{n'} u_{n'}(r, \mathbf{\Omega}) \equiv \psi_b(r, \mathbf{\Omega}), \text{ for } r \in \partial V \text{ and } \mathbf{\Omega} \cdot \mathbf{n} < 0.
\]

(8)

Multiplying Eq. (8) by the basis functions, \( u_n \), and integrating over external surfaces and incoming directions results in

\[
\sum_{n'} \tilde{\psi}_{n'} \oint_{\partial V} \int_{\Omega \cdot n < 0} u_n u_{n'} \, d\Omega ds \equiv \oint_{\partial V} \int_{\Omega \cdot n < 0} u_n \psi_b \, d\Omega ds,
\]

(9)

which may be multiplied by a penalty parameter, \( \alpha \), and added to Eq. (5) resulting in
\[
\sum_{n'} \bar{\psi}_{n'} [\langle \Omega \cdot \nabla u_n, \Omega \cdot \nabla u_{n'} \rangle + \langle G u_n, \Omega \cdot \nabla u_{n'} \rangle + \langle \nabla u_n, \Omega \cdot \nabla u_{n'} \rangle + \langle \nabla u_n, G u_{n'} \rangle + \alpha \oint_{\Omega} \int_{\Omega < 0} u_n u_{n'} d\Omega ds]
\]
\[
= \langle \Omega \cdot \nabla u_n, Q \rangle + \langle G u_n, Q \rangle + \alpha \oint_{\Omega} \int_{\Omega < 0} u_n \psi \, d\Omega ds, \text{ for all } n.
\]

3. SPHERICAL HARMONICS INTEGRALS

For the P_n approximation in SCEPTRE, the angular dependence of the angular flux is approximated by the spherical harmonics
\[
u_n(\mathbf{r}, \Omega) = q_n(\mathbf{r})\tilde{Y}_{l,m}(\Omega) = \varphi_n(\mathbf{r}) \tilde{Y}_{\text{mom}}(\Omega),
\]
where \(\tilde{Y}_{l,m}(\Omega)\) are the normalized, real (sin/cos) spherical harmonics, and the \(l,m\) indices are combined to a single index, \(\text{mom}\). The angular integrals for the various moments needed in the least-squares approximation, Eq. (5), are pre-computed using recursion relations described in great detail in Ref. [10]. After expanding the angular dependence of the angular flux in the spherical harmonics, the second-order term in Eq. (5) is
\[
\langle \Omega \cdot \nabla u_n, \Omega \cdot \nabla u_{n'} \rangle = \sum_{i=1}^{D} \sum_{i'=1}^{D} \frac{\partial \varphi_n}{\partial \xi_i} \frac{\partial \varphi_{n'}}{\partial \xi_i'} \langle \mu_i \mu_{i'} \tilde{Y}_{\text{mom}} \tilde{Y}_{\text{mom}'} \rangle,
\]
where \(D\) is the dimensionality of the problem (\(D=1, 2\) or 3), \(\xi_i\) are the Cartesian coordinates (x, y, or z), and \(\mu_i\) are the direction cosines (\(\mu_x, \mu_y\) or \(\mu_z\)). The first-order terms from Eq. (5) become
\[
\langle \Omega \cdot \nabla u_n, \nabla u_{n'} \rangle + \langle G u_n, \Omega \cdot \nabla u_{n'} \rangle
\]
\[
= (\sigma_t - \sigma_{t'}) \sum_{i=1}^{D} \frac{\partial \varphi_n}{\partial \xi_i} \varphi_{n'} \langle \mu_i \tilde{Y}_{\text{mom}} \tilde{Y}_{\text{mom}'} \rangle
\]
\[
+ (\sigma_t - \sigma_{t'}) \sum_{i=1}^{D} \varphi_n \frac{\partial \varphi_{n'}}{\partial \xi_i} \langle \mu_{i'} \tilde{Y}_{\text{mom}} \tilde{Y}_{\text{mom}'} \rangle
\]
and the zero-order term becomes
\[
\langle G u_n, G u_{n'} \rangle = (\sigma_t - \sigma_i)(\sigma_t - \sigma_{i'}) \varphi_n \varphi_{n'}
\]
The required spherical-harmonics moments needed for the LS P_n solver (and also for the other second-order P_n solvers) are, therefore
The moments integrals computed from the recursion relations are tested and demonstrated to be accurate to an error tolerance of $10^{-12}$ for spherical harmonics of order up to $l=64$.

4. LEBEDEV QUADRATURE

Unlike the full-range angular integration of the volume integral terms described in the previous section, the external boundary integral terms require half-range angular integration, and therefore, cannot be pre-computed and are integrated numerically. Furthermore, the half-range integrations generally require a higher-order quadrature integration than full-range integrations. For this reason, the Lebedev quadrature [11], which provides quadrature weights and directions for integrals up to order 130, has been implemented in SCEPTRE. The Lebedev quadrature implementation has been tested up to order 130 by comparing analytic full-range integrals of polynomials up to order 130 with numerically-calculated values to an error tolerance of $10^{-12}$. A high-order Lebedev quadrature integration is then used to compute the half-range boundary integrals in Eqs. (7) and (10).

4.1. Lebedev Galerkin Quadrature

The SCEPTRE code contains both $S_n$ and $P_n$ solvers in a single framework, allowing the user to mix and match on a group-by-group basis between the various solvers, based on the physics of the problem of interest. In order to facilitate the handoff of source data between the $S_n$ and $P_n$ solvers, a Galerkin quadrature [12] is used so that accuracy is not degraded in mapping between the two angular spaces.

A Galerkin quadrature has the property that the moment-to-discrete (M2D) and discrete-to-moment (D2M) matrices are inverses of each other, requiring that there be an equal number of directions and moments and that the matrices be non-singular. For a level-symmetric angular quadrature there is a simple procedure for specifying which spherical-harmonics moments to use for a given quadrature order to construct a non-singular M2D [12]. For Lebedev quadrature, the situation is different. For a given quadrature order, a Lebedev quadrature set has about one-half to one-third of the number of directions as the same-order level-symmetric set. Therefore, the Lebedev Galerkin set requires many fewer spherical harmonics to construct a non-singular M2D than the comparable level-symmetric set. A Gram-Schmidt procedure is used to construct a set of linearly-independent column vectors for a Galerkin moment-to-discrete matrix from potential candidates, as sketched in Fig. 1.

The procedure starts with a set of column vectors, $\hat{v}_k$, where the $k^{th}$ vector is the $k^{th}$ spherical harmonic, $\bar{Y}_k$, evaluated at the discrete directions, $\Omega_d$, in the quadrature set under consideration. The specific sequence of spherical harmonics in the set depends upon the dimensionality of the problem geometry. $\hat{v}_0 = \bar{Y}_0(\Omega_d)$ becomes the first column in the M2D matrix. Subsequent columns in the M2D matrix are determined by sequentially testing the remaining column vectors, $\hat{v}_k$, for linear independence against the column vectors that have been added to the M2D matrix. Vectors that are not linearly independent with the vectors in the partially-constructed M2D
matrix are discarded, while others are added to the M2D matrix. The procedure is continued until the number of columns in the M2D matrix is equal to the number of rows (i.e. the number of directions in the quadrature set is equal to the moments). The Galerkin moments have been tested up to order 64 by comparing the M2D times D2M with the identity matrix.

Figure 1. Gram-Schmidt procedure for finding linearly independent Galerkin moments.
5. RESULTS

The LS $P_n$ and $S_n$ solvers are applied to test problem [9], consisting of an isolated isotropic source region surrounded by a void and reflector, as shown in Fig. 2, with reflective bottom and left boundaries and vacuum top and right.

![Coarse triangular mesh for the test problem. A uniform isotropic source is located in the lower-left region, and the non-void regions have isotropic scattering with cross sections of $\sigma_f=0.2$, $\sigma_s=0.19$.](image)

Figs. 3a-d show the scalar flux distribution for the test problem from SCEPTRE LS $S_n$ and $P_n$ calculations. The $P_8$ results (Fig. 3a) are substantially too low in the source region but are qualitatively quite good away from the source (as compared with the $S_{30}$ and $P_{30}$ results). The $S_8$ results (Fig. 3b) are much better than the $P_8$ results in the source region but are dominated by ray effects away from the source. The $P_{30}$ and $S_{30}$ results (Figs. 3c and 3d) qualitatively agree well with each other. Fig. 4 compares the SCEPTRE $P_{30}$ and $S_{30}$ results with the MCNP results provided in Ref. [9], for a path through the reflector region showing good qualitative agreement.

Fig. 5 compares the two methods for imposing vacuum boundary conditions in the LS $P_n$ method. The figure shows the percentage difference between the penalty-method results and the divergence-theorem method, for penalty factors from 0.01 to 100, for the average scalar flux in the source region and for the global leakage. The results are fairly insensitive to the value of the penalty factor, the results being nearly identical for a penalty factor of 1.
The value picked for the penalty parameter, $\alpha$, tends modify the spatial distribution of the error, i.e. a larger value for $\alpha$ should tend to decrease the error in the vicinity of the boundary at the expense of the solution in other regions and conversely for a smaller value of $\alpha$. The particular method chosen for handling the boundary conditions and the specific value chosen for $\alpha$ is problem dependent, but generally if the main quantity of interest is a boundary value such as a leakage current, the penalty method with a large penalty parameter may be the best choice, and otherwise the Green's function method may be better, but this warrants further investigation.

Figure 3a. Scalar flux distribution for test problem: SCEPTRE LS $P_8$ transport.

Figure 3b. Scalar flux distribution for test problem: SCEPTRE LS $S_8$ transport.

Figure 3c. Scalar flux distribution for test problem: SCEPTRE LS $P_{30}$ transport.

Figure 3d. Scalar flux distribution for test problem: SCEPTRE LS $S_{30}$ transport.
Convergence results for the test problem are shown in Tables Ia and Ib for the average scalar flux in the source region and for the average global leakage. A Taylor expansion of the method indicates 2nd-order convergence for linear-triangular finite elements, away from boundaries and interfaces. The observed convergence rates are generally lower than expected, except for the average leakage for the Sn method, where the convergence rate, p, approaches 4. The convergence results are obtained by uniformly refining the mesh shown in Fig. 2, and then using Richardson extrapolation to obtain a converged solution using the apparent observed convergence rate to obtain the Richardson extrapolated value.

Table Ia. LS test problem
P16 spatial convergence

<table>
<thead>
<tr>
<th>Mesh Refinement Level</th>
<th>Convergence Rate $p$</th>
<th>$\bar{\phi}_{\text{source}}$</th>
<th>$\bar{J}_{\text{leakage}}$</th>
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Table Ib. LS test problem
S16 spatial convergence

<table>
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<th>Mesh Refinement Level</th>
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5.1. Convergence Results for a One-Dimensional Test Problem

In order to further investigate the observed convergence rates for the SCEPTRE LS algorithms, a one-dimensional test problem is devised that is analogous to the 2D test problem in the previous section. The coarse mesh for the test problem is shown in Fig. 6.

Figure 6. Coarse mesh for 1D test problem. Analogous to the 2D test problem, a uniform isotropic source is located in the left region, and the non-void regions have isotropic scattering with cross sections of $\sigma_t=0.2$, $\sigma_s=0.19$.

For uniform refinements of the coarse mesh, the SCEPTRE convergence results for a LS P$_{16}$ and S$_{16}$ calculation are shown in Tables IIa and IIb. The results are compared with Richardson extrapolated values, showing convergence to the expected rate of $p=2$ in all cases.

Table IIa. LS 1D test problem P$_{16}$ spatial convergence

<table>
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<th>Mesh Refinement Level</th>
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Table IIb. LS 1D test problem S$_{16}$ spatial convergence

<table>
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</table>

Apparently, the lower-than-expected observed convergence rates are a multidimensional effect, consistent with a comment in p. 238 of Roach [13], "Also, note that in general 2-D and 3-D problems, spatial cross derivative truncations terms arise that can profoundly affect the character and magnitude of the solution." This behavior is under investigation.
3. CONCLUSIONS

The SCEPTRE LS $P_n$ and $S_n$ algorithms have been described, including two different methods for imposing vacuum boundary conditions in the $P_n$ solver. The results show qualitatively good agreement with each other and with published results, though the convergence rates for a 2D test problem have deviated from the expected theoretical values. The two methods for imposing vacuum boundary conditions in the $P_n$ solver are compared, showing that the results are relatively insensitive to the value of the penalty factor chosen. The moments integrals needed for the $P_n$ solver are described, and also the Lebedev quadrature implementation for computing the half-range boundary integrals for the $P_n$ method are described. Work is in progress to understand the lower than expected convergence rates of the LS algorithms in the source region.

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