CALCULATING ADJOINT FLUXES IN THE CODE DRAGON USING THE COLLISION PROBABILITY METHOD

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ABSTRACT

In neutron transport theory, computation of adjoint and generalized adjoint fluxes may present some difficulties, particularly when the collision probability solution technique is considered. This paper proposes a simple method to compute these adjoint fluxes. By defining a pseudo adjoint flux, one can apply an algorithm, similar to that required for the evaluation of the direct neutron flux, to adjoint flux calculations. We will also show that the multigroup iterative solution procedure used in DRAGON, and required due to the presence of the scattering source, can be easily modified in such a way that the performance of the solution algorithm is preserved for the adjoint problem.

1. INTRODUCTION

The first mention of the adjoint neutron flux in neutronics dates from the introduction of a perturbations methods which can be used to compute variations in k_{eff} resulting form small changes in cross sections.[1] Later, Gandini introduced the generalized adjoint problem as a mean to evaluate, using perturbation theory, variations in functionals of the flux which could be used for sensitivity and optimisation studies.[2] Since then, the range of problems that have been analyzed using this technique has increased substantially.[3] Accordingly, the development of efficient adjoint fluxes computation methods remains a major concern.

In most cases, the perturbation theory method has been applied only to problems where the algorithm required to compute the adjoint flux is very similar to that required for the direct flux. This is the case when the one velocity homogeneous neutron transport equation is considered or

for the multigroup neutron diffusion equation in general geometry. On the other hand, adjoint flux calculations for the heterogeneous and multigroup transport problem have been scarce.

Here we will develop an algorithm which can be used to solve the adjoint transport equation using the collision probability (CP) technique. The first step in this development is to transform the adjoint CP equations into a form which is compatible with the direct form of these equations. Then, we will discuss how the multigroup iterative solution algorithm can be modified to ensure that the solution of the adjoint flux equations becomes as efficient as the current solution to the direct problem.

In this paper, after describing the general concepts associated with the adjoint flux, we will discuss how the CP technique can be used to solve the adjoint transport equation. We will then present in Section 4 the modifications to the CP multigroup solution algorithm that can be used to accelerate the adjoint flux solution problem. In section 5 we will compare the performance of the direct and adjoint solution algorithm. Finally, we will conclude.

2. THEORY

In the absence of external source the transport equation may be written in the form:

$$(\mathbf{A} - \lambda \mathbf{F})\, \phi(\vec{r}, \vec{\Omega}, E) = 0 \tag{1}$$

were $\lambda = 1/k_{eff}$ is the eigenvalue of the problem. To the direct equation we may associate the following adjoint equation:[1,4]

$$(\mathbf{A}^* - \lambda^* \mathbf{F}^*)\, \phi^*(\vec{r}, \vec{\Omega}, E) = 0 \tag{2}$$

Here $\phi^*(\vec{r}, \vec{\Omega}, E)$ is known as the adjoint neutron flux. The relation between any direct operator $\mathbf{X}$ ($\mathbf{A}$ or $\mathbf{F}$) and its adjoint $\mathbf{X}^*$ is based on the scalar product conservation, namely, we will require $\mathbf{X}$ and $\mathbf{X}^*$ to satisfy:

$$\Big\langle f(\vec{r}, \vec{\Omega}, E), \mathbf{X}\; h(\vec{r}, \vec{\Omega}, E) \Big\rangle = \Big\langle \mathbf{X}^* f(\vec{r}, \vec{\Omega}, E), h(\vec{r}, \vec{\Omega}, E) \Big\rangle \tag{3}$$

where the scalar product is defined as:

$$\Big\langle f(\vec{r}, \vec{\Omega}, E), h(\vec{r}, \vec{\Omega}, E) \Big\rangle = \int d^3r \int dE \int d^2\Omega\, f(\vec{r}, \vec{\Omega}, E) h(\vec{r}, \vec{\Omega}, E) \tag{4}$$

Note that the identity $\lambda^* = \lambda$ is a consequence of the above definition of adjoint operators.

For the direct flux problem, the boundary conditions to be imposed will be specified by the physics of the problem to be solved. On the other hand, determining the boundary conditions

associated with the adjoint transport problem is not as simple. Since in transport the operator $\mathbf{A}$ contains the differentiation operator $\vec{\nabla}$, evaluating $\mathbf{A}^*$ involves an integration by part of Eq. (4). This transformation introduces an additional surface integral involving both the direct and adjoint flux which must vanish. This last condition will be used to select boundary conditions for the adjoint flux which will be complimentary with those used for the direct flux.[4]

For a fixed source reactor, the direct and adjoint fluxes will satisfy the following equations:

$$\mathbf{M}\phi(\vec{r},\vec{\Omega},E) = S(\vec{r},\vec{\Omega},E) \tag{5}$$

$$\mathbf{M}^*\phi^*(\vec{r},\vec{\Omega},E) = S^*(\vec{r},\vec{\Omega},E) \tag{6}$$

where the operator $\mathbf{M}$ is defined as

$$\mathbf{M} = \mathbf{A} - \mathbf{F}$$

These equations should have a solution for any fixed direct or adjoint source. The adjoint sources are arbitrary and satisfy the identity:

$$\Big\langle \phi^*(\vec{r},\vec{\Omega},E), S(\vec{r},\vec{\Omega},E) \Big\rangle = \Big\langle S^*(\vec{r},\vec{\Omega},E), \phi(\vec{r},\vec{\Omega},E) \Big\rangle$$

We may also consider a generalized adjoint problem, where the generalized adjoint $\Gamma^*(\vec{r},\vec{\Omega},E)$ satisfies the following equation:

$$\left(\mathbf{A}^* - \lambda\mathbf{F}^*\right)\Gamma^*(\vec{r},\vec{\Omega},E) = S^*(\vec{r},\vec{\Omega},E) \tag{7}$$

where λ is the eigenvalue of Eq. (1). The generalized adjoint sources are still arbitrary provided they satisfy the relation (Eqs. (1) and (7)):

$$\Big\langle S^*(\vec{r},\vec{\Omega},E), \phi(\vec{r},\vec{\Omega},E) \Big\rangle = 0 \tag{8}$$

Note that the generalized adjoint is not uniquely determined since it can include a component proportional to the adjoint flux:

$$\Gamma^*(\vec{r},\vec{\Omega},E) \equiv \Gamma^*(\vec{r},\vec{\Omega},E) + \alpha\phi^*(\vec{r},\vec{\Omega},E) \tag{9}$$

However, if one imposes the following orthogonality relation on the generalized adjoint:[3]

$$\Big\langle \Gamma^*(\vec{r},\vec{\Omega},E), \mathbf{F}\phi(\vec{r},\vec{\Omega},E) \Big\rangle = 0 \tag{10}$$

then Γ^* in Eq. (9) can be uniquely determined by imposing:

$$\alpha = -\frac{\left\langle \Gamma^*(\vec{r},\vec{\Omega},E), \mathbf{F}\phi(\vec{r},\vec{\Omega},E) \right\rangle}{\left\langle \phi^*(\vec{r},\vec{\Omega},E), \mathbf{F}\phi(\vec{r},\vec{\Omega},E) \right\rangle} \tag{11}$$

3. APPLICATION TO THE COLLISION PROBABILITY METHOD

The application of the CP method to the integral transport equation involves a spatial discretization of the multigroup scalar flux. As a result, instead of having to deal with a distribution $f(\vec{r},\vec{\Omega},E)$ in the evaluation of the scalar product, we will work with a discretized distribution f_i^g defined as:

$$f_i^g = \frac{1}{V_i}\int_{V_i} d^3r \int_{E_g}^{E_{g-1}} dE \int_{4\pi} d^2\Omega\, f(\vec{r},\vec{\Omega},E)$$

The generalized scalar product in Eq. (4) then takes the form

$$\left\langle f(\vec{r},\vec{\Omega},E), h(\vec{r},\vec{\Omega},E) \right\rangle = \left\langle \vec{f},\vec{h} \right\rangle = \sum_{g=1}^{G}\sum_{i=1}^{N} f_i^g\, V_i\, h_i^g = \left(\vec{f}\right)^{\mathrm{T}} \mathbf{V}\vec{h} \tag{12}$$

Here the vectors $\vec{f}$ and $\vec{h}$, which are discretized over G energy groups and N regions, are defined as:

$$\vec{f} = \begin{pmatrix} \vec{f}^{\,1} \\ \vec{f}^{\,2} \\ \vdots \\ \vec{f}^{\,G} \end{pmatrix} \qquad \text{with} \qquad \vec{f}^{\,g} = \begin{pmatrix} f_1^g \\ f_2^g \\ \vdots \\ f_N^g \end{pmatrix}$$

while $\mathbf{V}$ is a diagonal matrix containing the regional volumes.

Using Eqs. (3) and (12), the relation between an operator and its adjoint will then be given by :

$$\mathbf{X}^* = \mathbf{V}^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{V} \tag{13}$$

3.1 THE EIGENVALUE PROBLEM

The CP discretization of the multigroup integral neutron transport equation in the absence of external sources can be written in the form:[5]

$$\vec{\phi} = \mathbf{P_{vv}}\mathbf{R}_\lambda\vec{\phi} \tag{14}$$

where $\vec{\phi}$ is the multigroup and multiregion average neutron flux while $\mathbf{P_{vv}}$ and $\mathbf{R}_\lambda$ are the multigroup collision probability, and production matrices respectively. Here $\mathbf{R}_\lambda$ is defined in terms of the scattering ($\Sigma_\mathbf{s}$), the fission ($\boldsymbol{\nu}\Sigma_\mathbf{f}$) and the spectrum ($\boldsymbol{\chi}$) matrices as follows:

$$\mathbf{R}_\lambda = \Sigma_\mathbf{s} + \frac{1}{k_{\mathit{eff}}}\boldsymbol{\chi}\left(\boldsymbol{\nu}\Sigma_\mathbf{f}\right)^\mathrm{T}$$

where $\lambda = 1/k_{\mathit{eff}}$ is the eigenvalue of the problem and the collision probability matrix in Eq. (14) already takes into account the boundary conditions.[5]

Using Eqs. (13) and (14) and the fact that the matrices $\mathbf{R}_\lambda$ and $\mathbf{V}$ are commutative, we can write the CP equation associated with the adjoint flux in the form:

$$\vec{\phi}^* = \left(\mathbf{R}_\lambda\right)^\mathrm{T}\mathbf{P_{vv}}\vec{\phi}^* \tag{15}$$

where we have used the fact that the collision probability matrix verifies:[6]

$$\left(\mathbf{P_{vv}}\right)^\mathrm{T}\mathbf{V} = \mathbf{V}\mathbf{P_{vv}}$$

One immediately sees that the numerical algorithms required to solve Eqs. (14) and (15) are somewhat different. One way to by-pass this problem is to rewrite Eq. (15) in terms of a pseudo-adjoint flux $\vec{\phi}^+ = \mathbf{P_{vv}}\vec{\phi}^*$:

$$\vec{\phi}^+ = \mathbf{P_{vv}}\left(\mathbf{R}_\lambda\right)^\mathrm{T}\vec{\phi}^+ \tag{16}$$

Eqs. (14) and (16) are now identical apart from the transposition of the production matrix. In addition, the adjoint flux can be obtained directly from $\vec{\phi}^+$ using:

$$\vec{\phi}^* = \left(\mathbf{R}_\lambda\right)^\mathrm{T}\vec{\phi}^+$$

3.2 THE FIXED SOURCE PROBLEM

In the case where external sources $\vec{Q}_e$ are also present in the transport equation, Eq. (14) becomes:[5]

$$\vec{\phi} = \mathbf{P_{vv}}\mathbf{R}\vec{\phi} + \mathbf{P_{vv}}\vec{Q}_e = \mathbf{P_{vv}}\mathbf{R}\vec{\phi} + \vec{S} \tag{17}$$

where the matrix $\mathbf{R}$ then takes the form:

$$\mathbf{R} = \Sigma_\mathbf{s} + \boldsymbol{\chi}\left(\boldsymbol{\nu}\Sigma_\mathbf{f}\right)^\mathrm{T}$$

For a given adjoint source $\vec{S}^*$ the adjoint equation will be:

$$\vec{\phi}^* = (\mathbf{R})^{\mathrm{T}} \mathbf{P_{vv}} \vec{\phi}^* + \vec{S}^* \tag{18}$$

Once again we may rewrite the equation in term of the pseudo adjoint flux to obtain:

$$\vec{\phi}^+ = \mathbf{P_{vv}} (\mathbf{R})^{\mathrm{T}} \vec{\phi}^+ + \mathbf{P_{vv}} \vec{S}^* \tag{19}$$

Eqs. (17) and (19) are now identical apart from the fixed sources and the production matrix. In addition, the adjoint flux can be obtained directly from $\vec{\phi}^+$ using:

$$\vec{\phi}^* = (\mathbf{R})^{\mathrm{T}} \vec{\phi}^+ + \vec{S}^*$$

3.3 THE GENERALIZED ADJOINT PROBLEM

For the generalized adjoint problem with source $\vec{S}^*$, the generalized adjoint $\vec{\Gamma}^*$ will be a solution of:

$$\vec{\Gamma}^* = (\mathbf{R}_\lambda)^{\mathrm{T}} \mathbf{P_{vv}} \vec{\Gamma}^* + \vec{S}^* \tag{20}$$

This equation is a fixed source problem with λ imposed to the eigenvalue associated with the direct flux problem in the absence of sources (see Eq. (14)).

Again this equation can be cast into a standard collision probability form using the technique described above. First one defines the pseudo generalized adjoint as:

$$\vec{\Gamma}^+ = \mathbf{P_{vv}} \vec{\Gamma}^* \tag{21}$$

The CP equation then becomes:

$$\vec{\Gamma}^+ = \mathbf{P_{vv}} (\mathbf{R}_\lambda)^{\mathrm{T}} \vec{\Gamma}^+ + \mathbf{P_{vv}} \vec{S}^* \tag{22}$$

Once $\vec{\Gamma}^+$ is known, it can then be used to built the generalized adjoint according to:

$$\vec{\Gamma}^* = (\mathbf{R}_\lambda)^{\mathrm{T}} \vec{\Gamma}^+ + \vec{S}^* \tag{23}$$

As we discussed in Section 2, there is an additional orthogonalization constraint imposed on the generalized adjoint flux solution, namely, it should satisfy Eq. (10), which may now be rewritten as:

$$\left\langle \vec{\Gamma}^+, \boldsymbol{\chi} (\boldsymbol{\nu}\Sigma_{\mathbf{f}})^{\mathrm{T}} \vec{\phi} \right\rangle = 0$$

This constraint can be ensured by using the following flux normalization scheme for the generalized adjoint flux:

$$\vec{\Gamma}_N^+ = \vec{\Gamma}^+ + \alpha\vec{\phi}^+$$

where $\vec{\Gamma}^+$ is the solution of Eq. (22),

$$\alpha = -\frac{\left\langle \vec{\Gamma}^+, \boldsymbol{\chi}\left(\boldsymbol{\nu}\Sigma_{\mathbf{f}}\right)^{\mathrm{T}}\vec{\phi}\right\rangle}{\left\langle \vec{\phi}^+, \boldsymbol{\chi}\left(\boldsymbol{\nu}\Sigma_{\mathbf{f}}\right)^{\mathrm{T}}\vec{\phi}\right\rangle} \tag{24}$$

and $\vec{\Gamma}_N^+$ is the required orthogonalized generalized adjoint flux.

Finally, let us recall that in perturbation theory, the adjoint is generally used to compute eigenvalue variations. On the other hand, the generalized adjoint is used to approximate functionals variations (reaction rate ratio for example) in such a way that the generalized adjoint sources are associated with the corresponding functional derivative.[3]

4. COMPUTING DIRECT AND ADJOINT FLUXES IN DRAGON

4.1 THE EIGENVALUE PROBLEM

The numerical algorithm used in DRAGON to solve the collision probability equations for the neutron flux is based on the following decomposition of Eq. (14):[7]

$$\vec{\phi} = \mathbf{W_{vv}}\left(\mathbf{L}+\mathbf{U}+\frac{1}{k_{\mathit{eff}}}\boldsymbol{\chi}\left(\boldsymbol{\nu}\Sigma_{\mathbf{f}}\right)^{\mathrm{T}}\right)\vec{\phi} = \mathbf{W_{vv}}\left(\mathbf{L}+\mathbf{U}\right)\vec{\phi} + \mathbf{W_{vv}}\vec{Q}_f \tag{25}$$

where $\mathbf{L}$ and $\mathbf{U}$ are lower and upper triangular matrices which contain the down- and up-scattering elements of $\Sigma_{\mathbf{s}}$ respectively. The diagonal elements $\mathbf{D}$ of the scattering matrix are already combined with the collision probability matrix to form the matrix $\mathbf{W_{vv}}$:

$$\mathbf{W_{vv}} = \left(\mathbf{I}-\mathbf{P_{vv}D}\right)^{-1}\mathbf{P_{vv}}$$

The solution method then involves two different levels of iteration: namely, the power (or outer) iteration which deals with the fission source and the multigroup (inner) iteration which deals with the scattering sources (assuming a fixed fission source). The outer iteration is only affected by the direct and adjoint fission sources which have a similar form provided one defines $\boldsymbol{\chi}^* = \boldsymbol{\nu}\Sigma_{\mathbf{f}}$ and $(\boldsymbol{\nu}\Sigma_f)^* = \boldsymbol{\chi}$.

We will now concentrate our discussion on the inner iteration. In the case where $\mathbf{U} \equiv 0$, the solution $\vec{\phi}$ can be obtained directly (without iterations) using

$$\vec{\phi} = \mathbf{W_{vv}L}\vec{\phi} + \mathbf{W_{vv}}\vec{Q}_f \tag{26}$$

provided the solution algorithm starts with $g = 1$ and ends with $g = G$.

When $\mathbf{U}$ has non-zero components only for the groups $g \geq H$, one still solves the transport equation directly using Eq. (26) for $g = 1$ to $g = H - 1$. For the remaining groups, an iterative process of the form

$$\vec{\phi}(k+1) = \mathbf{W_{vv}}\mathbf{L}\vec{\phi}(k+1) + \mathbf{W_{vv}}\mathbf{U}\vec{\phi}(k) + \mathbf{W_{vv}}\vec{Q}_f \tag{27}$$

is required.

When solving the pseudo-adjoint CP equations, the main difference lies in the fact that one has to deal with the transpose of the scattering source. This means that the up- and down-scattering matrices become the adjoint down- and up-scattering matrices respectively. As a result, one could define $\mathbf{L}^+ = \mathbf{U}^{\mathrm{T}}$ and $\mathbf{U}^+ = \mathbf{L}^{\mathrm{T}}$. If the group ordering is inverted, the iterative solution algorithm described above will involve all the groups without taking into account the specific properties of the adjoint down-scattering matrix.

Here we will describe the modifications to this algorithm that are required for the adjoint flux solution to explicitly take into account the properties of the up-scattering (or adjoint down-scattering) matrix. The first step consist in inverting the group solution order, namely we will solve the adjoint flux equation starting with $g = G$ and finishing at $g = H$ using an iterative process similar to that described in Eq. (27). Here we will use

$$\vec{\phi}^+(k+1) = \mathbf{W_{vv}}\mathbf{U}^+\vec{\phi}^+(k+1) + \mathbf{W_{vv}}\mathbf{L}^+\vec{\phi}^+(k) + \mathbf{W_{vv}}\vec{Q}_f^+ \tag{28}$$

before directly solving for the remaining groups ($g = H - 1$ to $g = 1$) using

$$\vec{\phi}^+ = \mathbf{W_{vv}}\mathbf{U}^+\vec{\phi}^+ + \mathbf{W_{vv}}\vec{Q}_f^+ \tag{29}$$

Note that for the case where the up-scattering matrix vanishes, both the direct and adjoint flux solution can be obtained without the need for the inner iteration.

We have modified the flux solution module of the DRAGON code in such a way that it can now deal with both the direct and pseudo-adjoint equations according to the technique described above. The general structure of the solution algorithm is thereby preserved and remains compatible with the already programmed acceleration techniques. Hence, the efficiency of the direct and pseudo-adjoint flux computation algorithm should be similar.

4.2 OTHER CASES

Formally the algorithm for an eigenvalue and for a fixed source problem are identical, except that in the later case the eigenvalue is fixed and equal to one and the fixed source term in the inner iterations has to take into account the external source ($\vec{Q}_e$):

$$\vec{Q}_f = \boldsymbol{\chi} \left(\boldsymbol{\nu}\Sigma_{\mathbf{f}}\right)^{\mathrm{T}} \vec{\phi} + \vec{Q}_e$$

The generalized adjoint flux computation is very close to a fixed source flux calculation, apart from the eigenvalue which is fixed and equal to the critical problem eigenvalue. In order to prevent numerical contamination from the adjoint flux we renormalize the flux using Eq. (24) after each outer iteration.

5. RESULTS

In order to compare the performance of the direct and pseudo-adjoint flux computation algorithms in DRAGON we first analyzed a two group four region small cell problem. The properties of our cell are similar to those provided in Reference 8 except that here the fuel region is further subdivided into two subregions containing different mixtures. The results we obtained

Table I. Direct and adjoint flux calculations for the modified small cell problem

Flux	Direct		Adjoint		Generalized adjoint	
($\mathrm{cm}^{-2}.s^{-1}$)	group 1	group 2	group 1	group 2	group 1	group 2
Region 1	3.0500	0.5114	0.2406	0.6541	-0.0184	0.4392
Region 2	3.0283	0.5321	0.2410	0.6598	-0.0162	0.4339
Region 3	2.9959	0.5514	0.1751	0.3187	-0.0379	0.3040
Region 4	2.9760	0.5651	0.2876	1.5439	0.0207	1.8692
k_{eff}	1.18529		1.18529		1.18529 (imposed)	
Total inner iterations	6		4		9	

for the direct and adjoint flux problems are presented in Table I. As expected, the computed k_{eff} for both problems are identical. One can also observe that the total number of iterations required for the adjoint flux solution is smaller than that required for the direct flux solution. This just reflects the fact that the pseudo-adjoint flux equation is better conditioned in this case.

Table II. Generalized adjoint source for the small cell problem

Source (cm^{-1})	Region 1	Region 2	Region 3	Region 4
Group 1	-0.01866	-0.01728	-0.03907	-0.01027
Group 2	0.02194	0.00034	-0.03867	0.15110

Here because the scattering matrix $\Sigma_{\mathbf{s}}$ is full, we could, by transposition and group order inversion, associate to the adjoint problem an equivalent direct flux problem. This new set of problems was also solved independently. As one expect, the new direct solution converges in 4 inner iterations toward the initial adjoint flux solution. Similarly, the new adjoint solution converges in 6 inner iterations toward the initial direct flux solution.

For the generalized adjoint computation we used the sources provided in Table II. Note that some of these sources are negatives as expected since the generalized source should satisfy Eq. (8). The convergence rate of the generalized adjoint solution is somewhat degraded with respect to both the direct and adjoint problem (see Table I). One might expect this effect to be the result of a large increase in the number of outer iterations since such a behaviour is observed when a fixed source problem is solved for cells containing fissile regions. However, the single additional outer iterations required in this case does not justify this interpretation. The degradation in the convergence rate here is mainly due to the fission source initialization options used in DRAGON for the iterative solution procedure. For eigenvalue problems, the initial fission sources are computed using a flat flux distribution inside the cell. On the other hand, for fixed source problems, the fission source is initialized to 0.0 in the first outer iteration. As a result fission sources contributions are only taken into account after the first outer iteration has been completed thereby increasing only slightly the total number of inner iterations.

Table III. Direct and adjoint flux calculations for the CANDU cell problem

	Direct	Adjoint	Generalized adjoint
k_{eff}	0.97571	0.97572	0.97572 (imposed)
Total inner iterations	18	21	25
One group flux calculations	859	1007	1153

Finally we analyzed a 2-D CANDU fuel channel and a 17x17 PWR assembly.[9] These calculations were performed using 69-group cross sections taken from the WIMS-AECL Winfrith library (26 fast groups).[10] The generalized adjoint sources in these cases were derived from a functional representing the ratio of the total reaction rate in the cell to the integrated flux.[3]

The results we obtained for the CANDU cell calculations are presented in Table III. As one can see the direct and adjoint problems give results for k_{eff} which are within the convergence criterion imposed on the iterative process (10^{-5}). The total number of inner iterations is somewhat different because the form of the up-scattering matrix $\mathbf{U}$ favors the group ordering used in the direct solution rather than that used in the adjoint solution. The number of distinct one group flux calculation required in the adjoint solution while being 17 % larger than that required for the direct problem still represents only 70% of the total number of calculations that is required when the problem is simulated using the original direct flux algorithm (1449 one group flux calculations).

Table IV. Direct and adjoint calculations for the PWR 17x17 assembly

	Direct	Adjoint	Generalized Adjoint
k_{eff}	1.259546	1.259547	1.259547 (imposed)
Total inner iterations	14	12	18
One group flux calculations	644	594	878

For the PWR assembly, the results are presented in Table IV. Again the direct and adjoint k_{eff}

are in very good agreement. In this case the total number of inner iterations for the direct and adjoint solution are almost identical and the up-scattering matrix only favors slightly the adjoint solution. The number of distinct one group flux calculation required in the adjoint solution is 8 % lower than that required for the direct problem and represents again 72 % of the total number of calculations required with the original algorithm (828 one group flux calculations).

Finally, as it was the case for the small cell problem, the generalized adjoint problem always requires more one group flux calculations than either the direct or adjoint problem to converge. The major effect here is again the differences in the fission source initialization scheme for the outer iteration as opposed to major changes in the number of outer iterations.

CONCLUSION

We showed that the adjoint and generalized adjoint CP equations can be modified to a form which can be solved using an algorithm similar to that used for the solution of the direct CP equations. We also proposed modifications to the multigroup iterative CP solution procedure that acknowledge the properties of the adjoint down-scattering matrix. As a result, the performance of the direct and adjoint flux solution algorithm programmed in DRAGON are now equivalent.

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