A “Mildly Inconsistent” Method for Accelerating Upstream Corner Balance Transport in Slab Geometry

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1 Abstract

A new method for accelerating the Upstream Corner Balance (UCB) [1] discretization of the Boltzmann Transport Equation is introduced. The inconsistent acceleration equations for the UCB discretization are derived by applying the “Modified 4-Step” diffusion synthetic acceleration technique [2] *not* to the UCB discretization, but instead to the simple corner balance (SCB) transport discretization. Wareing [3] has previously demonstrated that mildly inconsistent modified 4-step DSA works very well for orthogonal-mesh transport differencings. The convergence properties of the new method have been determined by Fourier analysis and this data agrees well with the observed effectiveness. Our results indicate that this inconsistent acceleration scheme will greatly increase the rate of iterative convergence of UCB compared to that of source iteration alone.

2 Introduction

During the past few years there as been an increased effort to devise robust transport differencings for unstructured meshes, specifically arbitrarily connected grids of polygons. UCB, developed by Adams [1] was designed with unstructured polygonal meshes in mind. The viability of UCB as a transport scheme depends sensitively on the availability of techniques to rapidly iterate the equations to convergence, especially for optically thick and diffusive problems. One method that has been evaluated for accelerating the iterative convergence rate of UCB is Transport Synthetic Acceleration (TSA) [4]. TSA uses a reduced quadrature order S_N equation as its low-order acceleration operator. This paper details another acceleration technique which uses SCB derived “Modified 4-Step” diffusion synthetic acceleration equations to accelerate the convergence of UCB transport iterations. This approach was observed to be effective by Palmer [5] in $r-z$ geometry, but to this date the convergence properties of the method have not been analyzed.

3 Description

3.1 Upstream Corner Balance

The Upstream Corner Balance (UCB) [1] discretization of the Boltzmann transport equation was developed after analyzing the characteristics of the Simple Corner Balance discretization (SCB) [1]. The SCB technique imposes particle balance over subcell grid volumes, and closure relationships which provide additional constraints to make the problem well-posed. In slab geometry, the SCB equations for the two half-cells are:

$$\frac{2\mu}{\Delta x}(\psi_{m,i} - \psi_{m,i-1/2}) + \sigma_{t,i,L}\psi_{m,i,L} = \frac{1}{2}\sigma_{s,i,L}\phi_{i,L} + \frac{1}{2}S_{i,L} \tag{1}$$

$$\frac{2\mu}{\Delta x}(\psi_{m,i+1/2} - \psi_{m,i}) + \sigma_{t,i,R}\psi_{m,i,R} = \frac{1}{2}\sigma_{s,i,R}\phi_{i,R} + \frac{1}{2}S_{i,R} \tag{2}$$

We have assumed isotropic scattering, an isotropic source and mono-energetic particles. We require closure relationships for the cell-center ($\psi_{m,i}$) and cell-edge ($\psi_{m,i\pm1/2}$) unknowns. SCB chooses an upstream closure for the cell-edge unknown, namely:

$$\left.\begin{array}{l}\psi_{m,i+1/2} = \psi_{m,i,R} \\ \psi_{m,i-1/2} = \psi_{m,i-1,R}\end{array}\right\} \quad , \quad \mu > 0 \quad , \tag{3}$$

$$\left.\begin{array}{l}\psi_{m,i+1/2} = \psi_{m,i+1,L} \\ \psi_{m,i-1/2} = \psi_{m,i,L}\end{array}\right\} \quad , \quad \mu < 0 \quad . \tag{4}$$

The cell-center angular flux is replaced by a *simple* average of the angular fluxes in the left and right half-cells:

$$\psi_{m,i} = \left(\frac{\psi_{m,i,L} + \psi_{m,i,R}}{2}\right)\left\{\begin{array}{ll}\mu > 0 & , \\ \mu < 0 & .\end{array}\right. \tag{5}$$

This definition of the cell-centered flux couples the left and right half cell fluxes in both of the balance equations. This requires the solution of a 2×2 linear system in each zone for slab geometry transport sweeps. On general 2D polygons, the transport sweep requires the inversion of an $N \times N$ matrix in each spatial cell where N is the number of vertices of each polygon. Inverting an $N \times N$ matrix for the solution within each spatial cell becomes very expensive as the number of vertices increases. What would be desirable is a discretization method that maintains the favorable properties of SCB but eliminates the need to invert an $N \times N$ matrix for each cell. Adams has designed a modification to SCB that alleviates this inefficiency by replacing Eq. (5) with an *upstream* closure eliminating the $N \times N$ inversion:

$$\psi_{m,i} = \left\{\begin{array}{ll}\psi_{m,i,L} + \frac{1}{4}\left[(\frac{\sigma_s\phi+S}{\sigma_t})_{i,R} - (\frac{\sigma_s\phi+S}{\sigma_t})_{i,L}\right] + \beta(\tau_{m,i,L})(\psi_{m,i,L} - \psi_{m,i-1/2}) & , \quad \mu > 0 \quad , \\ \psi_{m,i,R} + \frac{1}{4}\left[(\frac{\sigma_s\phi+S}{\sigma_t})_{i,L} - (\frac{\sigma_s\phi+S}{\sigma_t})_{i,R}\right] + \beta(\tau_{m,i,R})(\psi_{m,iR} - \psi_{m,i+1/2}) & , \quad \mu < 0 \quad .\end{array}\right. \tag{6}$$

This closure eliminates the need for a matrix inversion in each zone. Furthermore, Eq. (6) helps UCB to limit to the same (or somewhat more accurate) discretized diffusion equation for optically thick and diffusive problems. This closure also provides improved accuracy for optically thin and intermediate regimes.

3.2 Modified Four-Step Diffusion-Synthetic Acceleration

The UCB method, while very well suited for resolving solutions in the thick diffusion limit, suffers from the same problem that plagues all solution methods that rely on source iteration: In an infinite, homogeneous medium the spectral radius, ρ, the factor by which the error is reduced from one iteration to the next, is equal to the scattering ratio, c. Thus, in large systems with $c \approx 1$, the method will take a prohibitively long time to converge on the true solution of the system. Diffusion Synthetic Acceleration (DSA) can be used to alleviate this problem and has been shown to unconditionally reduce the slab geometry spectral radius from c to $c/3$ (for isotropic scattering),provided the discretized diffusion equations are consistent with the discretized transport equations [2]. Adams and Martin have derived consistent DSA equations for discontinuous finite element transport equations using a technique which has been called “Modified 4Step”.

3.2.1 Deriving the Modified 4-Step DSA Equations

To derive the Modified 4-Step DSA equations the following steps are taken:

- Take the 0^{th} angular moment of SCB transport equations (1-5).
- Take the 1^{st} angular moment of SCB transport equations (1-5).
- Subtract the resulting equations from the converged system to obtain a new system of equations for the additive corrections to the scalar flux resulting from source iteration.
- Make the within-cell approximation to the cell-edge scalar flux correction.
- Eliminate all currents in favor of the left and right half-cell flux corrections.

The resulting discontinuous diffusion equations are in terms of just the left and right additive flux corrections. These equations can be written in matrix form as a banded seven-stripe matrix or as follows:

$$-\frac{1}{2}(\frac{1}{3\sigma_{t,i}})(\frac{f_{i,R}^{\ell+1}-f_{i,L}^{\ell+1}}{\Delta x_i}) - \frac{\gamma}{2}(f_{i-1,R}^{\ell+1} - f_{i,L}^{\ell+1}) + \frac{1}{2}(\frac{1}{3\sigma_{t,i-1}})(\frac{f_{i-1,R}^{\ell+1}-f_{i-1,L}^{\ell+1}}{\Delta x_{i-1}}) + \frac{\sigma_{a,i}\Delta x_i}{2} f_{i,L}^{\ell+1} = \frac{\sigma_{s,i}\Delta x_i}{2}(\phi_{i,L}^{\ell+\frac{1}{2}} - \phi_{i,L}^{\ell}) \quad , \tag{7}$$

and

$$\frac{1}{2}(\frac{1}{3\sigma_{t,i}})(\frac{f_{i,R}^{\ell+1}-f_{i,L}^{\ell+1}}{\Delta x_i}) + \frac{\gamma}{2}(f_{i,R}^{\ell+1} - f_{i+1,L}^{\ell+1}) - \frac{1}{2}(\frac{1}{3\sigma_{t,i+1}})(\frac{f_{i+1,R}^{\ell+1}-f_{i+1,L}^{\ell+1}}{\Delta x_{i+1}}) + \frac{\sigma_{a,i}\Delta x_i}{2} f_{i,R}^{\ell+1} = \frac{\sigma_{s,i}\Delta x_i}{2}(\phi_{i,R}^{\ell+\frac{1}{2}} - \phi_{i,R}^{\ell}) \quad , \tag{8}$$

where γ is the quadrature normalization defined as the sum of the quadrature weights:

$$\gamma = \sum_{n=1}^{N} w_n \tag{9}$$

and the left and right half-cell flux corrections are defined as:

$$f_{i,(L,R)}^{\ell+1} = \phi_{i,(L,R)}^{\ell+1} - \phi_{i,(L,R)}^{\ell+\frac{1}{2}} \quad . \tag{10}$$

The SCB method has much in common with discontinuous finite element dicretizations of the transport equation. In slab geometry, SCB is completely equivalent to the mass matrix lumped linear discontinuous transport equations. This motivated us to use Modified 4-Step DSA with SCB in order to produce the discontinuous acceleration equations. In contrast, the non-conventional form of the UCB closure, Eq. (6), makes deriving a set of discontinuous acceleration equations using Modified 4-Step DSA a difficult task.

3.3 A "Mildly Inconsistent" Technique for Accelerating UCB

Since applying the modified four-step DSA scheme to UCB becomes algebraically complex in higher spatial dimensions and non-Cartesian geometries, we have devised another way. Because UCB and SCB can be made to have very similar or *identical* asymptotic diffusion limits, we conjecture that the Modified 4-Step DSA corrections calculated for one could be used to accelerate the other. In this *mildly inconsistent* manner the SCB derived modified four-step DSA correction is used to accelerate the UCB discretization. Wareing, Walters and Morel [3] demonstrated that it was possible to use the acceleration equations derived from bilinear-discontinuous transport to accelerate the iterative convergence of a linear-bilinear nodal transport differencing scheme. This result has guided us in our work.

4 Results

A Fourier analysis was performed on the UCB discretization using SCB-derived DSA corrections. The unaccelerated UCB discretization had a spectral radius, as expected, equal to the scattering ratio, c. The SCB-accelerated UCB discretization has a maximum spectral radius of 0.2950, for $c = 0.999999$ and an optical thickness of approximately 0.93 mean-free-paths (mfp). Also of note is the expected thin limit value of approximately 0.2247 at 0.01 mfp. The plot also shows a spectral radius approaching zero for very optically thick cells. We have implemented this acceleration technique in a slab geometry UCB transport code. Numerical results were generated for slabs with 100 cells, an S_{32} quadrature set, vacuum boundary conditions, zero sources and a convergence to $\phi = 0$ for a range of optical thicknesses matching that of the Fourier analysis. To get the most accurate results all of the Fourier modes were excited by picking a random initial guess for the angular flux. The problem was then allowed to run until a stable spectral radius was achieved. Figure (1) shows that our implementation code agrees well with our analysis: the SCB-derived DSA effectively accelerates the UCB discretization.

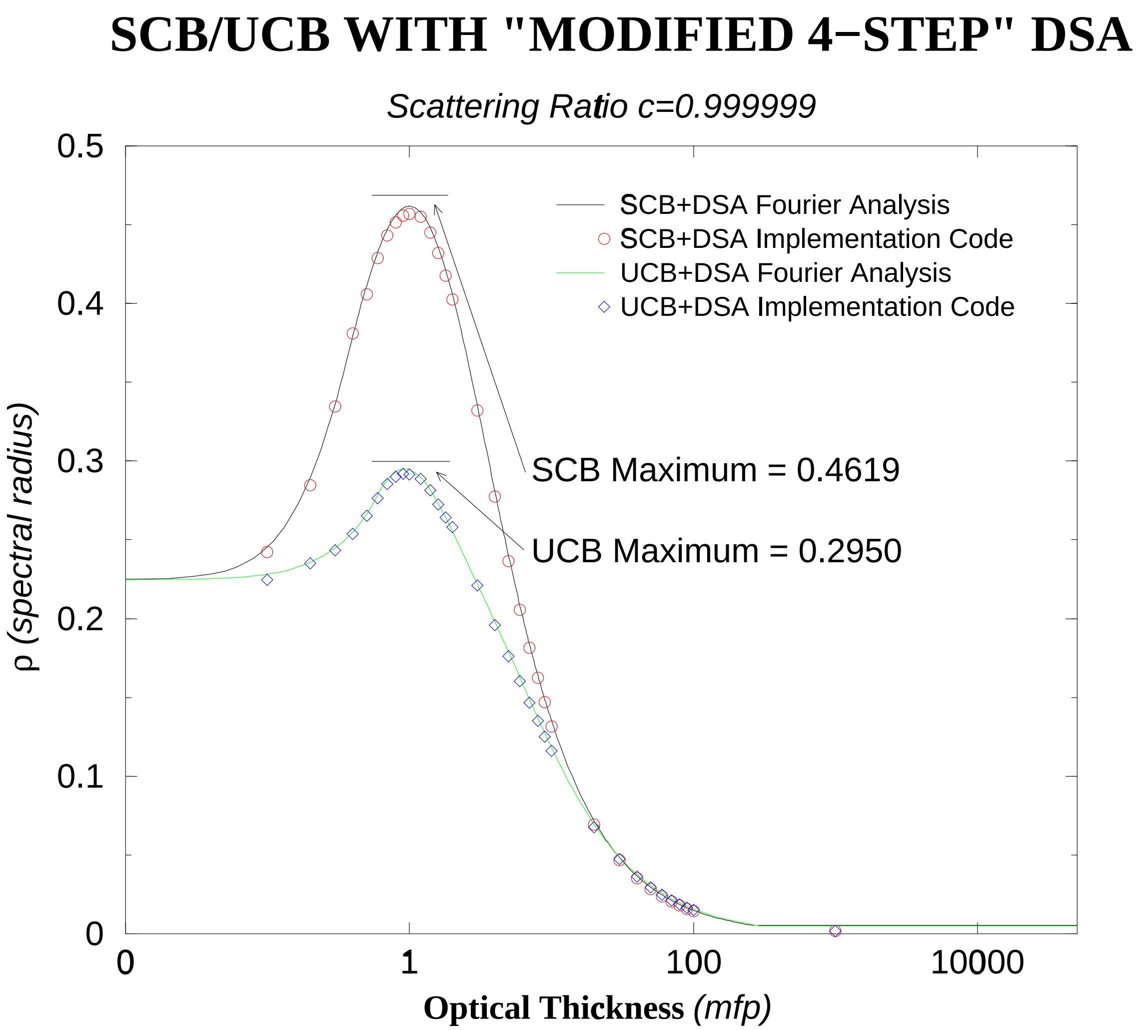

Figure 1: Fourier mode analysis of the UCB discretization accelerated inconsistently with Modified 4-Step DSA derived from SCB.

5 Conclusion

We have introduced a new method for accelerating the slab geometry UCB discretization of the transport equation. To validate the method a Fourier analysis was performed and an implementation code was written. The results of this analysis show that the SCB-derived acceleration rapidly increases the rate of iterative convergence. Since the Modified 4-Step DSA equations can be derived from multidimensional and curvilinear geometry SCB equations [5], and, provided that the fore-mentioned acceleration equations can be efficiently solved, our results offer hope that the acceleration of UCB can be accomplished.

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References

[1] Marvin L. Adams, "Subcell Balance Methods for Radiative Transfer on Arbitrary Grids," *Transport Theory and Statistical Physics*, **26**, 385 (1997).

[2] M.L. Adams and W.R. Martin, "Diffusion Synthetic Acceleration of Discontinuous Finite Element Transport Iterations," *Nuclear Science and Engineering*, **111**, 145 (1992).

[3] T.A. Wareing, W.F. Walters and J.E. Morel, "Diffusion-Accelerated Solution of the Two-Dimensional X-Y Sn Equations with Linear-Bilinear Nodal Differencing," *Nuclear Science and Engineering*, **118**, 122 (1994).

[4] G.L. Ramone and M.L Adams, "A Transport Synthetic Acceleration Method for Transport Iterations," *Nuclear Science and Engineering*, **125**, 257 (1997).

[5] Todd S. Palmer, "Curvilinear Geometry Transport Discretizations in Thick Diffusive Regions," Doctoral Thesis (University of Michigan), Lawrence Livermore National Laboratory, UCRL-ID-114256, June (1993).