A UNIFIED THEORY OF ZERO AND POWER REACTOR NOISE VIA A MASTER EQUATION APPROACH

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ABSTRACT

Fluctuations of the neutron flux in a multiplicative system can be divided into two categories, namely zero noise and power reactor noise. As the name indicates, they dominate (i.e. are observable) at different power levels. In addition, they are described via different mathematical tools, namely master equations and the Langevin equation, respectively. Because of the different mathematical methods used in the description of the two noise types, there is no joint or unified description that treats a case when both types of noise are present concurrently.

The subject of the present paper is to develop a unified theory of zero and power reactor noise through calculating the probability distribution of the neutrons in a core with fluctuating material properties. A backward type master equation formalism is used with point kinetics, and the fluctuating cross sections are represented by a binary pseudorandom process. A solution is obtained which is significantly more complicated than the cases of zero noise or power reactor noise separately, which are also given in the paper. It is shown that the general solution contains both the zero noise and the power reactor noise in the sense that the two forms can be extracted individually as limiting cases of the general solution.

1. INTRODUCTION

Traditionally, two major types of neutron fluctuations are considered in multiplicative systems. In reactors at low power and constant material properties, such as a zero power reactor (critical assembly), the reason of the neutron fluctuations is the inherently random characteristics of neutron transport. This type of noise is called zero reactor noise or zero noise in short. The fluctuations in such systems carry information about some nuclear properties of the system, including reactivity. Hence, methods were elaborated to use such fluctuations for determining subcritical reactivity. The two main such methods are the Feynman-alpha and Rossi-alpha methods, respectively[1-4]. The theory of such fluctuations is based on master equations of stochastic branching processes[5-6]. One property of the zero noise is that its variance is proportional to the static (mean) neutron flux or average power of the system.

The second type of neutron fluctuations, also called power reactor noise, exist in large power producing reactors. It is induced by the fluctuations of the reactor materials, such as boiling of the coolant, vibrations of control rods and the core barrel, temperature fluctuations in pressurized water reactors etc. These fluctuations are described by stochastic differential equations for the neutron fluctuations directly (as opposed to the probability distribution as in the master equations of zero noise) with coefficients as random processes (the fluctuating cross sections). This technique is also called the Langevin equation or Langevin technique, and the results are expressed in form of auto- and cross-spectra of the neutron fluctuations. These fluctuations are used for extracting information on the perturbations (cross section fluctuations) that generate the neutron noise. Changes in the noise can indicate anomalous changes in system state or the appearance of new anomalies. Thus the use of power reactor noise for diagnostic purposes is also called neutron noise diagnostics or reactor diagnostics. The variance or the amplitude of the spectra of power reactor noise is proportional to the square of the static noise or static reactor power. Hence, as the name also indicates, power reactor noise dominates in power producing reactors with high power (and where fluctuations of the cross-spectra are also likely to occur), whereas zero noise dominates in low power systems (where, in addition, cross-section fluctuations are usually absent).

Due to the above reasons, i.e. different regions of validity and different mathematical tools, zero noise and power reactor noise have developed as two separate branches of science. This is a pragmatic approach that avoids to handle complications that are unimportant for practical applications. However, from the academical point of view, it is highly desirable to establish direct contact between these two branches of neutron fluctuations in form of a unified theory from which the zero noise and the power reactor noise components can be obtained as limiting cases. Then the neglections made when considering the zero and power reactor noise separately can be estimated. Such a unified theory will also illuminate the reason for the advantage of using the Langevin approach to the calculation of the power reactor noise, and should be able to describe the neutron noise in intermediate cases, where neither the zero reactor nor the power reactor noise dominates.

The development and application to a simple case of such a unified theory is the subject of the present paper. After a brief summary of the traditional derivation of the zero noise and power reactor noise formulas separately in a simple model case with master equations and Langevin equations respectively, the unified model is introduced. A simple binary pseudorandom model of the cross section fluctuations is chosen, and a backward type master equation is derived for the probability distribution of the neutron number in the system. This approach contains both the zero noise and power reactor noise components. From the equations for the full probability distribution, equations for the two first moments are derived and solved. The solution for the second moment is then compared to those obtained from the pure zero reactor and power reactor models. It is shown that those two solutions can be reproduced from the general solution of the unified model. However, as can be expected, the limits of zero power and high power do not yield the former zero and power reactor components exactly, only in the cases of vanishing cross section fluctuations and large cross section fluctuations, respectively. Thus the results show the effect of co-existence of the two types of noise sources on the induced noise very clearly. The treatment also shows that the equations of the unified model are much more involved than the traditional zero noise and power reactor noise equations, and hence it also illustrates why it is practical to use the Langevin approach for the latter.

2. TRADITIONAL ZERO AND POWER REACTOR NOISE THEORY

In the following we shall use a rather simplified reactor model, which is unnecessarily restrictive as long as the two types of noise are treated separately. It will be however necessary to use the simple model in the general case which is substantially more complicated. Hence the simple model will also be used in this section so that the results of traditional theory and the unified theory will become comparable.

We shall thus use an infinite homogeneous system with a static subcriticality ρ and with an external source S. One-speed theory will be used for the temporal dependence of the neutron density in the system, and thus both space and energy dependence will be neglected. In addition, delayed neutrons will not be taken into account, i.e. all neutrons are considered as prompt neutrons. The probabilities of fission, capture and absorption will be described by the one-group parameters

$$\lambda_f = v\Sigma_f; \quad \lambda_c = v\Sigma_c; \quad \lambda_a = \lambda_f + \lambda_c \tag{1}$$

With the above parameters the reactivity and prompt neutron generation time ρ and Λ are introduced as usual as

$$\rho = \frac{\nu\lambda_f - \lambda_a}{\nu\lambda_f}; \qquad \Lambda = \frac{1}{\nu\lambda_f} \tag{2}$$

2.1 CALCULATION OF THE ZERO NOISE

Zero noise is described by master equations, that can be either of the forward or the backward type. Master equations define probability distributions of various dimensions (one-time, two-time etc.). In order to compare with power reactor noise, one needs power spectra that can be calculated from temporal correlations, thus we need two-point (two-time) distributions. As discussed e.g. in reference 7, when calculating correlations, it is more advantageous to use the backward formalism. In addition, the general theory can also be easier formulated by using the backward approach. Thus the backward equations will also be used in this paper.

The calculations with the backward approach go in two steps. First the distribution of the number of neutrons, or its generating function, due to one single source neutron is derived. Second, a formula is derived that connects the generating function of the single-particle induced distribution with that induced by a continuous stream of (random) source particles. This formula is often called the Bartlett-formula. From the asymptotic form $(t \to \infty)$ of the Bartlett formula, moments of the stationary distribution can be derived. We shall follow this path below.

For the correlations one needs the single-particle induced probability distribution

$$P(N_1, t;\ N_2, t+\tau) \tag{3}$$

as the probability of finding N_1 neutrons at t and N_2 neutrons at $t+\tau$, due to one neutron starting the process at $t = 0$. We introduce also the probability generating function of P as

$$G(x_1, t; x_2, t+\tau) = \sum_{N_1}\sum_{N_2} x_1^{N_1} x_2^{N_2} P(N_1, t;\ N_2, t+\tau) \tag{4}$$

Without going into details (these can be found e.g. in reference 7) we quote the backward master equation that can be derived for G:

$$\begin{aligned}\frac{d}{dt}G(x_1, t; x_2, t+\tau) &= \lambda_f \sum_n p(n) G^n(x_1, t; x_2, t+\tau) \\ &\quad + \lambda_c - \lambda_a G(x_1, t; x_2, t+\tau)\end{aligned} \tag{5}$$

where $p(n)$ is the number distribution of source neutrons. Defining the first moment of the neutron number as

$$N(t) \equiv \langle N(t) \rangle = \left.\frac{\partial G(x_1, t; x_2, t+\tau)}{\partial x_1}\right|_{x_1 = x_2 = 1}, \tag{6}$$

one obtains from Eq. (5)

$$\frac{dN(t)}{dt} = \nu\lambda_f N(t) - \lambda_a N(t) + \delta(t) = \frac{\rho}{\Lambda}N(t) + \delta(t) \tag{7}$$

where

$$\nu \equiv \sum_n n p(n) \tag{8}$$

and the initial condition

$$N(0) = 1 \tag{9}$$

was added in form of a delta function. With a Laplace transform one obtains from Eq. (7)

$$N(s) = \frac{1}{s - \frac{\rho}{\Lambda}} \tag{10}$$

and

$$N(t) = e^{\frac{\rho}{\Lambda}t} \tag{11}$$

For the second moment

$$M_{NN}(t,\tau) \equiv \langle N(t)N(t+\tau)\rangle = \left.\frac{\partial^2 G(x_1, t; x_2, t+\tau)}{\partial x_1 \partial x_2}\right|_{x_1 = x_2 = 1} \tag{12}$$

one obtains

$$\frac{dM_{NN}(t,\tau)}{dt} = \nu\lambda_f M_{NN}(t,\tau) - \lambda_a M_{NN}(t,\tau) + \lambda_f \langle \nu\ (\nu - 1)\rangle N(t)N(t+\tau) \tag{13}$$

Here

$$\langle \nu\ (\nu - 1)\rangle \equiv \sum_n n(n-1)p(n) \tag{14}$$

Defining

$$Q_{NN}(t,\tau) = \lambda_f \langle \nu\ (\nu - 1)\rangle N(t)N(t+\tau) \tag{15}$$

Eq. (13) can be compactly written as

$$\frac{dM_{NN}(t,\tau)}{dt} = \frac{\rho}{\Lambda} M_{NN}(t,\tau) + Q_{NN}(t,\tau) \tag{16}$$

Assuming

$$M_{NN}(0,\tau) = 0 \qquad \text{for } \tau > 0 \tag{17}$$

(which, as discussed in reference 7, is true for the correlation of the detected neutrons and not for the correlation of the neutron number, but we shall use this in the present work), from a comparison of (7) and (16) one obtains

$$M_{NN}(t,\tau) = \int_0^t N(t-t')Q_{NN}(t',\tau)dt' \tag{18}$$

As it will be seen soon, for the evaluation of the correlation of the source-induced distribution, one does not need to evaluate this integral, only an integral over the source $Q_{NN}(t,\tau)$.

In the next step one defines the probability distribution of the neutrons induced by a source S that was switched on at $t = 0$,

$$\tilde{P}(N_1, t;\ N_2, t+\tau) \tag{19}$$

and its generating function

$$\tilde{G}(x_1, t;\ x_2, t+\tau) \tag{20}$$

The relationship between $\tilde{G}$ and G is given by the Bartlett-formula[8-10] as

$$\tilde{G}(x_1, t;\ x_2, t+\tau) = e^{S\int_0^t [G(x_1, t;\ x_2, t+\tau) - 1]dt} \tag{21}$$

From (21) asymptotic values of the first and second moment can be obtained as follows. For

$$N_0 \equiv \lim_{t\to\infty} \langle \tilde{N}(t) \rangle = \lim_{t\to\infty} \left. \frac{\partial \tilde{G}(x_1, t;\ x_2, t+\tau)}{\partial x_1} \right|_{x_1 = x_2 = 1} \tag{22}$$

one obtains with simple algebra

$$N_0 = \frac{S\Lambda}{-\rho} \tag{23}$$

For the second moment one defines the covariance function or the correlation function of the fluctuations as

$$\begin{aligned} C_{NN}(\tau) &\equiv \lim_{t\to\infty} \langle \tilde{N}(t)\tilde{N}(t+\tau) \rangle - \langle \tilde{N}(t) \rangle \langle \tilde{N}(t+\tau) \rangle \\ &= \langle (\tilde{N}(t) - N_0)(\tilde{N}(t+\tau) - N_0) \rangle \equiv \langle \delta N(t) \delta N(t+\tau) \rangle \end{aligned} \tag{24}$$

From (22) one obtains

$$C_{NN}(\tau) = N_0 \int_0^\infty Q_{NN}(t',\tau)dt' \tag{25}$$

and with further simple algebra

$$C_{NN}(\tau) = \lambda_f \langle \nu\,(\nu - 1) \rangle N_0 \frac{\Lambda}{(-2\rho)} e^{\frac{\rho}{\Lambda}|\tau|} \tag{26}$$

From (26) one obtains for the power spectrum of $\delta N(t)$

$$APSD_{\delta N}(\omega) = \int_{-\infty}^{\infty} C_{NN}(\tau) e^{-i\omega\tau} d\tau \tag{27}$$

the result

$$APSD_{\delta N}(\omega) = \lambda_f^{-1} \frac{\langle \nu\,(\nu - 1) \rangle}{\nu^2} N_0 |G_0(\omega)|^2 \tag{28}$$

Here,

$$G_0(\omega) = \frac{1}{i\omega\Lambda - \rho} \tag{29}$$

is the zero reactor transfer function without delayed neutrons for a subcritical reactor.

It is (28) which needs to be reproduced in the solution of the general theory in the low power limit or when the cross section fluctuations are absent. It is seen that the power spectrum of zero noise depends on N_0 linearly. Otherwise the spectrum is similar to that of power reactor noise induced by a white noise perturbation, as will be seen in the next section.

2.2 CALCULATION OF POWER REACTOR NOISE

Here one starts immediately with an equation for the neutron density, $N(t)$, as a stochastic process. Although in power reactor noise studies it is customary to start with a critical system, here in order to be compatible with the zero noise calculations and with the general theory, we shall assume a subcritical system with a source just as in the previous subsection. The reason is that zero noise in a critical system is not stationary, with a time dependent variance that diverges.

The static equation thus reads as

$$(\nu\lambda_f - \lambda_a) N_0 + S = \frac{\rho}{\Lambda} N_0 + S = 0 \tag{30}$$

from which one obtains the same result as in the foregoing,

$$N_0 = \frac{S\Lambda}{-\rho} \tag{31}$$

The neutron fluctuations are induced by fluctuations of some of the cross sections, and we shall assume that it is the absorption cross section that fluctuates. Then both the neutron density as well as λ_a will be time dependent. The time dependent point kinetic equation then reads as

$$\frac{dN(t)}{dt} = (\nu\lambda_f - \lambda_a(t))N(t) + S = \frac{\rho(t)}{\Lambda}N(t) + S \tag{32}$$

One now splits the time-dependent quantities into steady-sate values and fluctuations as

$$N(t) = N_0 + \delta N(t) \tag{33}$$

$$\lambda_a(t) = \lambda_a + \delta\lambda_a(t) = \lambda_a + \frac{\delta\rho(t)}{\Lambda} \tag{34}$$

Putting these into (32), the usual procedure is to subtract the static equations and neglect the second order term $\delta\rho(t)\delta N(t)$ to obtain

$$\frac{d\delta N(t)}{dt} = \frac{\delta\rho(t)}{\Lambda}N_0 + \frac{\rho}{\Lambda}\delta N(t) \tag{35}$$

and

$$\delta N(\omega) = \frac{N_0\delta\rho(\omega)}{i\omega\Lambda - \rho} \tag{36}$$

From here, with a formal application of the Wiener-Khinchin theorem one obtains for the power spectrum

$$APSD_{\delta N}(\omega) = N_0^2 \cdot APSD_{\delta\rho}(\omega)\left|G_0(\omega)\right|^2 \tag{37}$$

The function $G_0(\omega)$ is the same here as in (28), i.e. it is given by (29). Eq. (37) is similar to the zero noise formula (28), the two differences being the quadratic dependence on N_0, and an additional frequency dependence on the power spectrum of the reactivity fluctuations induced by the cross section fluctuations. The sum of these two expressions: (28) and (37), was often used to describe the situation in which both zero and power reactor noise is present. Eq. (37) is the second component that should be recovered as the limit of the general formula with high power or large cross section fluctuations.

It was also seen that technically, the Langevin technique of power reactor noise is much simpler than the master equation technique of the zero noise. This is due to the fact that in the Langevin formalism the zero noise component is missing. The noise is zero if there are no cross section fluctuations, as is seen e.g. from (37). In addition, the situation was also simplified by the linearisation, i.e. neglection of higher order terms (closure). Inclusion of the zero noise component when calculating the noise in systems with fluctuating cross sections requires the use of master equations, which will be achieved in the next section.

3. THEORY OF POWER REACTOR NOISE VIA MASTER EQUATIONS

3.1 THE MODEL OF THE CROSS SECTION FLUCTUATIONS

The static system will be described by the same static parameters as in the previous cases, as given by (1). The cross section fluctuations $\delta\lambda_a(t)$ will be assumed to be a simple pseudorandom binary process, i.e. $\delta\lambda_a(t)$ will take either the value α or $-\alpha$. At some discrete times, which follow Poisson statistics, the process will switch from state α to $-\alpha$ or vice versa. The probability of state change in dt is γdt. Thus the process will be described by the parameters α and γ.

As known, the auto-correlation and the power spectrum of the process $\alpha(t)$ is given as (see e.g. reference 11)

$$C(\tau) = \alpha^2 e^{-2\gamma|\tau|} \tag{38}$$

$$APSD_{\alpha}(\omega) = \frac{4\alpha^2\gamma}{4\gamma^2 + \omega^2} \tag{39}$$

It is seen from (38) and (39) that $\alpha(t)$ becomes a white noise (Wiener process) in the limit $\alpha^2 \to \infty$ and $\gamma \to \infty$ such that $\alpha^2 / \gamma = const$. Since

$$\alpha = \frac{\delta\rho}{\Lambda}, \tag{40}$$

for the auto spectrum of the reactivity perturbations $\delta\rho(t)$ represented by the process $\alpha(t)$ one has

$$APSD_{\delta\rho}(\omega) = \Lambda^2 \frac{4\alpha^2\gamma}{4\gamma^2 + \omega^2} \tag{41}$$

3.2 PROBABILITY DISTRIBUTION INDUCED BY A SINGLE PARTICLE

We shall start with a master equation for the joint probability distribution

$$P_{\beta}(N_1, \alpha_1, t;\; N_2, \alpha_2, t + \tau) \tag{42}$$

that there are N_1 neutrons in the medium of type α_1 at time t and N_2 neutrons in the medium of type α_2 at time $t + \tau$ in the system, induced by one initial neutron in the medium of type β at $t = 0$. β has alternative choices: either α or $-\alpha$.

Let us start with the initial medium of type α. For P_{α} a first-instant type master equation can be obtained by adding the probabilities of the mutually exclusive possibilities of having: (1) neither one collision nor the change of medium type within $(0, dt)$; (2) one collision of the initial neutron within $(0, dt)$; (3) one change of medium type within $(0, dt)$, respectively. One then obtains

$$\begin{aligned} & P_\alpha(N_1, \alpha_1, t;\ \ N_2, \alpha_2, t+\tau) \\ =&[1-\ (\lambda_a+\alpha+\gamma)dt\ ]P_\alpha(N_1, \alpha_1, t\text{-dt};\ \ N_2, \alpha_2, t\text{-dt}+\tau) \\ &+(\lambda_c+\alpha)dt\delta_{N_1, 0}\delta_{N_2, 0}P_\alpha(N_1, \alpha_1, t;\ \ N_2, \alpha_2, t+\tau) \\ &+\lambda_f dt\sum_l p(l)\times\prod_{i=1}^{l}P_\alpha(N_1^{(i)}, \alpha_1, t\text{-dt};\ N_2^{(i)}, \alpha_2, t\text{-dt}+\tau) \\ &+\gamma dtP_{-\alpha}(N_1, \alpha_1, t\text{-dt};\ \ N_2, \alpha_2, t\text{-dt}+\tau) \end{aligned} \tag{43}$$

The arguments of the products in Eq. (43) are subject to the following constraints:

$$\sum_{i=1}^{l} N_1{}^{(i)} = N_1; \qquad \sum_{i=1}^{l} N_2{}^{(i)} = N_2; \tag{44}$$

Introducing the generating function G_α as

$$G_\alpha(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau)=\sum_{\alpha_1, \alpha_2}\sum_{N_1, N_2} x_1{}^{N_1}x_2^{N_2}P_\alpha(N_1, \alpha_1, t;\ N_2, \alpha_2, t+\tau) \tag{45}$$

from Eq. (43), we obtain the following differential equation for the generating function G_α:

$$\begin{aligned} &\frac{d}{dt}G_\alpha(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau) = \lambda_c+\alpha \\ &-\ (\lambda_a+\alpha)G_\alpha(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau)+\gamma G_{-\alpha}(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau) \\ &+\lambda_f\sum_l p(l)G_\alpha^l(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau) \end{aligned} \tag{46}$$

which satisfies the initial condition

$$G_\alpha(x_1, \alpha_1, 0;\ x_2, \alpha_2, \tau) = x_1 \tag{47}$$

We shall now start deriving moments of this equation. For the first moment

$$N_\alpha(t)\equiv\langle N(t)\rangle = \left.\frac{\partial G_\alpha(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau)}{\partial x_1}\right|_{x_1=x_2=1}, \tag{48}$$

one obtains from Eq. (46)

$$\frac{dN_\alpha(t)}{dt} = (\frac{\rho}{\Lambda}-\alpha\text{ - }\gamma)N_\alpha(t)+\gamma N_{-\alpha}(t) \tag{49}$$

For the second moments, i.e. the correlations,

$$M_\alpha(t,\tau) \equiv \langle N(t)N(t+\tau)\rangle$$

$$= \left.\frac{\partial^2 G_\alpha(x_1, \alpha_1, t; x_2, \alpha_2, t+\tau)}{\partial x_1 \partial x_2}\right|_{x_1 = x_2 = 1} \tag{50}$$

one obtains from Eq. (46)

$$\frac{dM_\alpha(t,\tau)}{dt} = \left(\frac{\rho}{\Lambda} - \alpha - \gamma\right) M_\alpha(t,\tau) + \gamma M_{-\alpha}(t,\tau) + \lambda_f \langle \nu(\nu - 1)\rangle N_\alpha(t) N_\alpha(t+\tau) \tag{51}$$

with

$$M_\alpha(0,\tau) = 0 \tag{52}$$

As discussed in Section 2.1 on page 3, this initial condition corresponds to the correlations of the detected neutrons, which is the quantity that we calculated also in Section 2.1 on page 3.

For the initial medium of type $-\alpha$, similar equations can be obtained. Introducing the matrix,

$$X = \begin{bmatrix} \frac{\rho}{\Lambda} - \alpha - \gamma & \gamma \\ \gamma & \frac{\rho}{\Lambda} + \alpha - \gamma \end{bmatrix} \tag{53}$$

we obtain the following algebraic equations for the first and second moments:

$$\frac{d}{dt}\begin{bmatrix} N_\alpha \\ N_{-\alpha} \end{bmatrix} = X \begin{bmatrix} N_\alpha \\ N_{-\alpha} \end{bmatrix}, \qquad \begin{bmatrix} N_\alpha(0) \\ N_{-\alpha}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{54}$$

and

$$\frac{d}{dt}\begin{bmatrix} M_\alpha \\ M_{-\alpha} \end{bmatrix} = X \begin{bmatrix} M_\alpha \\ M_{-\alpha} \end{bmatrix} + \begin{bmatrix} Q_\alpha \\ Q_{-\alpha} \end{bmatrix}, \quad \begin{bmatrix} M_\alpha(0,\tau) \\ M_{-\alpha}(0,\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{55}$$

where, similarly to (15), the source of the second moment is defined as

$$Q_\alpha = \lambda_f \langle \nu(\nu - 1)\rangle N_\alpha(t) N_\alpha(t+\tau) \tag{56}$$

A temporal Laplace transform of Eq. (54) yields

$$N_\alpha(s) = \frac{s - \alpha + 2\gamma - \frac{\rho}{\Lambda}}{(s - s_1)(s - s_2)} \tag{57}$$

where s_1 and s_2 are the roots of the generalized inhour equation

$$H_\alpha(s) = s^2 - 2(-\gamma + \frac{\rho}{\Lambda})s - \alpha^2 - \gamma^2 + (-\gamma + \frac{\rho}{\Lambda})^2 . \tag{58}$$

The two roots s_1 and s_2 of (58) are given as

$$s_{1,2} = \frac{\rho}{\Lambda} - \gamma \mp \sqrt{\gamma^2 + \alpha^2} \tag{59}$$

This immediately shows that in order that a stationary solution should exist in the source-driven case, i.e. that s_2 also should be negative, an inequality between α and γ must be fulfilled in the form

$$\left(\frac{\rho}{\Lambda}\right)^2 - \frac{2\rho\gamma}{\Lambda} > \alpha^2 \tag{60}$$

Physically, this means that if γ tends to zero, α should be small enough such that the system is not supercritical with $-\alpha$. If γ is finite, the system can be supercritical, but for a given α, γ must be large enough that on the average, the time during which the system is supercritical between two subcritical regions is limited.

The inverse Laplace transform of (57) yields

$$N_\alpha(t) = \sum_{i=1}^{2} z_i e^{s_i t} \tag{61}$$

where

$$z_1 = \frac{\alpha - \gamma + \sqrt{\alpha^2 + \gamma^2}}{2\sqrt{\alpha^2 + \gamma^2}}, \quad z_2 = \frac{-\alpha + \gamma + \sqrt{\alpha^2 + \gamma^2}}{2\sqrt{\alpha^2 + \gamma^2}} \tag{62}$$

and the s_i are given in (59).

For the second moment, as will be seen shortly, we only need the Laplace transform

$$L(M_\alpha(t,\tau)) = \int_0^\infty M_\alpha(t,\tau) e^{-ts} dt , \tag{63}$$

which can be formally obtained from Eq. (55) by taking the Laplace transform. Thus we have

$$L(M_\alpha(t,\tau)) = \frac{L(Q_{-\alpha})\gamma + L(Q_\alpha)\left(s - \alpha + \gamma - \frac{\rho}{\Lambda}\right)}{H_\alpha(s)} , \tag{64}$$

where $L(Q_\alpha)$ is the Laplace transform of Q_α.

3.3 PROBABILITY DISTRIBUTION INDUCED BY SOURCE EMISSION

In the second step, we need to determine the source-induced probability distribution:

$$\tilde{P}_\beta(N_1, \alpha_1, t;\ N_2, \alpha_2, t+\tau) \tag{65}$$

i.e. the probability that there are N_1 neutrons in the medium of type α_1 at time t and N_2 neutrons in the medium of type α_2 at time $t+\tau$ in the system, induced by switching on the source in the medium of type β at $t = 0$.
This is achieved by the generalised Bartlett formula, which gives a relationship between the single-particle induced distribution (42) and the stationary source-induced distribution (65).

We start with the probability balance equation

$$\begin{aligned}
&\tilde{P}_\alpha(N_1, \alpha_1, t;\ N_2, \alpha_2, t+\tau)\\
&=[1-(S+\gamma)dt]\tilde{P}_\alpha(N_1, \alpha_1, t\text{-dt};\ N_2, \alpha_2, t\text{-dt}+\tau)\\
&+\gamma dt\ \tilde{P}_{-\alpha}(N_1, \alpha_1, t\text{-dt};\ N_2, \alpha_2, t\text{-dt}+\tau)\\
&+Sdt\sum_{n_1, n_2}\tilde{P}_\alpha(N_1-\mathrm{n}_1, \alpha_1, t\text{-dt};\ N_2-n_2, \alpha_2, t\text{-dt}+\tau)P_\alpha(\mathrm{n}_1, \alpha_1, t\text{-dt};\ n_2, \alpha_2, t\text{-dt}+\tau)
\end{aligned} \tag{66}$$

Similarly to the case of the single-particle induced distributions, the first moment $\tilde{N}_\alpha$ and the second moment $\tilde{M}_\alpha(t, \tau)$ of the generating function $\tilde{G}_\alpha$ can be defined. However the initial condition for the generating function will be different from that of the single-particle induced distribution, since no particle exists before the source is switched on at time $t = 0$:

$$\tilde{G}_\alpha(x_1, \alpha_1, 0;\ x_2, \alpha_2, \tau) = 1 \tag{67}$$

From Eq. (66), we obtain the following differential equation for the generating function $\tilde{G}_\alpha$:

$$\begin{aligned}
&\frac{d}{dt}\tilde{G}_\alpha(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau)\\
&=-(S+\gamma)\tilde{G}_\alpha(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau)+\gamma\tilde{G}_{-\alpha}(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau)\\
&+SG_\alpha(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau)\tilde{G}_\alpha(x_1, \alpha_1, t;\ x_2, \alpha_2, t+\tau)
\end{aligned} \tag{68}$$

For the initial medium of type $-\alpha$, a similar equation can be obtained. Introducing the matrix:

$$\tilde{X} = \begin{bmatrix} -S-\gamma+SG_\alpha & \gamma \\ \gamma & -S-\gamma+SG_{-\alpha} \end{bmatrix} \tag{69}$$

we obtain the algebraic equations for the generating functions:

$$\frac{d}{dt}\begin{bmatrix}\tilde{G}_{\alpha}\\ \tilde{G}_{-\alpha}\end{bmatrix} = \tilde{X}\begin{bmatrix}\tilde{G}_{\alpha}\\ \tilde{G}_{-\alpha}\end{bmatrix}, \qquad \begin{bmatrix}\tilde{G}_{\alpha}(0)\\ \tilde{G}_{-\alpha}(0)\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix} \tag{70}$$

The solution of Eq. (70) can be formally written as

$$\begin{bmatrix}\tilde{G}_{\alpha}(t)\\ \tilde{G}_{-\alpha}(t)\end{bmatrix} = exp\left(\int_0^t \tilde{X}dt'\right)\begin{bmatrix}1\\ 1\end{bmatrix} \tag{71}$$

It is also easy to show that

$$\tilde{N}_1 = \frac{\partial}{\partial x_1}\begin{bmatrix}\tilde{G}_{\alpha}(t)\\ \tilde{G}_{-\alpha}(t)\end{bmatrix}\Bigg|_{x_1 = x_2 = 1} = Se^{\gamma\Gamma t}\begin{bmatrix}\int_0^t N_{\alpha}(t')dt'\\ \int_0^t N_{-\alpha}(t')dt'\end{bmatrix}, \tag{72}$$

$$\tilde{N}_2 = \frac{\partial}{\partial x_2}\begin{bmatrix}\tilde{G}_{\alpha}(t)\\ \tilde{G}_{-\alpha}(t)\end{bmatrix}\Bigg|_{x_1 = x_2 = 1} = Se^{\gamma\Gamma t}\begin{bmatrix}\int_0^t N_{\alpha}(t'+\tau)dt'\\ \int_0^t N_{-\alpha}(t'+\tau)dt'\end{bmatrix}, \tag{73}$$

and

$$\tilde{M} = \frac{\partial^2}{\partial x_2\partial x_1}\begin{bmatrix}\tilde{G}_{\alpha}(t)\\ \tilde{G}_{-\alpha}(t)\end{bmatrix}\Bigg|_{x_1 = x_2 = 1} = S^2e^{\gamma\Gamma t}\begin{bmatrix}\int_0^t N_{\alpha}(t'+\tau)dt'\int_0^t N_{\alpha}(t')dt'\\ \int_0^t N_{-\alpha}(t'+\tau)dt'\int_0^t N_{-\alpha}(t')dt'\end{bmatrix} + Se^{\gamma\Gamma t}\begin{bmatrix}\int_0^t M_{\alpha}(t',\tau)dt'\\ \int_0^t M_{-\alpha}(t',\tau)dt'\end{bmatrix}, \tag{74}$$

where

$$\Gamma \equiv \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix} . \tag{75}$$

By Taylor expansion it is easy to see that

$$e^{\gamma\Gamma t} = I + \frac{1 - e^{-2\gamma t}}{2}\Gamma \rightarrow \frac{1}{2}I \text{ , as } t \rightarrow \infty \tag{76}$$

where I is the identity matrix.
From (57), (72) and (76), for the asymptotic first moment

$$\tilde{N}_0 \equiv \lim_{t \rightarrow \infty} \tilde{N}_\alpha(t) = \lim_{t \rightarrow \infty} \tilde{N}_{-\alpha}(t) \tag{77}$$

one obtains

$$\begin{aligned} \tilde{N}_0 &= \frac{S}{2}\left[\int_0^\infty N_\alpha(t')dt' + \int_0^\infty N_{-\alpha}(t')dt'\right] \\ &= \frac{S\Lambda}{-\rho} \cdot \frac{1}{1 + \frac{\alpha^2\Lambda^2}{(2\gamma\Lambda - \rho)\rho}} = \frac{N_0}{1 + \frac{\alpha^2\Lambda^2}{(2\gamma\Lambda - \rho)\rho}} > N_0 \end{aligned} \tag{78}$$

where N_0 is the steady-state neutron density in the system without cross section fluctuations. Thus the expected value is higher in the system with fluctuating properties. This deviation of the first moment from that in the unperturbed system cannot be seen in the Langevin approach due to the linearisation.

The asymptotic two-point second moments, i.e. the stationary value of the covariance or correlation function of the neutron numbers at t and $t + \tau$ is defined as

$$\tilde{C}_\alpha(\tau) = \lim_{t \rightarrow \infty} \langle \tilde{N}_\alpha(t)\tilde{N}_\alpha(t + \tau)\rangle - \langle \tilde{N}_\alpha(t)\rangle\langle \tilde{N}_\alpha(t + \tau)\rangle \tag{79}$$

From Eqns (72)-(79) one can derive an expression for the correlations function as

$$\begin{aligned} \tilde{C}_\alpha(\tau) &= \frac{S}{2}[L(M_\alpha(t, \tau))|_{s = 0} + L(M_{-\alpha}(t, \tau))|_{s = 0}] \\ &+ \frac{S^2}{4}[L(N_\alpha(t))|_{s = 0} - L(N_{-\alpha}(t))|_{s = 0}]\int_\tau^\infty [N_\alpha(t') - N_{-\alpha}(t')]\mathrm{dt'} \end{aligned} \tag{80}$$

where L stands for the Laplace transform.
By use of eqns (57), (61) and (64), we obtain the final expression of the auto-correlation function as

$$\tilde{C}(\tau) = -\frac{S^2\alpha^2}{(s_1 - s_2)s_1 s_2}\left(\frac{e^{s_1|\tau|}}{s_1} - \frac{e^{s_2|\tau|}}{s_2}\right)$$

$$+\frac{S\lambda_f\langle\nu(\nu-1)\rangle\left(2\gamma - \frac{\rho}{\Lambda}\right)}{(s_1^2 - s_2^2)s_1 s_2}\left(W_1\frac{e^{s_1|\tau|}}{s_1} - W_2\frac{e^{s_2|\tau|}}{s_2}\right), \tag{81}$$

where

$$W_1 = \alpha^2 + (\gamma - \sqrt{\alpha^2 + \gamma^2})\left(\gamma - \frac{\rho}{\Lambda}\right) \tag{82}$$

and

$$W_2 = \alpha^2 + (\gamma + \sqrt{\alpha^2 + \gamma^2})\left(\gamma - \frac{\rho}{\Lambda}\right). \tag{83}$$

3.4 AUTO-POWER SPECTRAL DENSITY AND DISCUSSION

From (81), the spectral density of the solution obtained in the unified model can be obtained as

$$APSD(\omega) = -\frac{2N_0^2\rho^2\alpha^2(s_1 + s_2)}{\Lambda^2 s_1 s_2(\omega^2 + s_2^2)(\omega^2 + s_1^2)}$$

$$+\frac{2N_0(-\rho)\lambda_f\langle\nu(\nu-1)\rangle\left(2\gamma - \frac{\rho}{\Lambda}\right)}{\Lambda(s_1^2 - s_2^2)s_1 s_2}\left(\frac{W_2}{\omega^2 + s_2^2} - \frac{W_1}{\omega^2 + s_1^2}\right) \tag{84}$$

It is seen that the power spectrum of the general model consists of two terms, one proportional to the first power of N_0 and one proportional to N_0^2. Thus these terms correspond to the zero noise and the power reactor noise, respectively. For the limits $N_0 \to 0$ and $N_0 \to \infty$ the zero and the power reactor noise terms become dominant, respectively. They are however not exactly equivalent to the traditional terms since the parameters included all contain the effect of both the cross section fluctuations and the stochastic transport effects of the zero noise. To see the equivalence with the pure zero and power reactor noise terms, we need to consider some special cases.

Case I. In the case of $\alpha = 0$, since $W_1 = 0$ and the first term is zero in Eq. (84), Eq. (84) becomes

$$APSD(\omega) = \frac{\lambda_f^{-1}\langle\nu(\nu-1)\rangle}{\nu^2}N_0\frac{1}{\Lambda^2\omega^2 + \rho^2}. \tag{85}$$

This expression is equivalent with the case of pure zero noise, Eq. (28).

Case II. In the case of $\frac{\alpha^2}{\gamma} = -\frac{2\rho}{\Lambda}$ and $\gamma \to \infty$, the second term goes to zero in (84). Further,

$$s_1 = \frac{\rho}{\Lambda} - \gamma - \sqrt{\alpha^2 + \gamma^2} = \frac{\rho}{\Lambda} - \gamma - \gamma\sqrt{1 + \frac{\alpha^2}{\gamma^2}} \approx -2\gamma, \tag{86}$$

$$s_2 = \frac{\rho}{\Lambda} - \gamma + \sqrt{\alpha^2 + \gamma^2} = \frac{\rho}{\Lambda} - \gamma + \gamma\sqrt{1 + \frac{\alpha^2}{\gamma^2}} \approx \frac{\rho}{\Lambda}, \tag{87}$$

therefore (84) becomes

$$APSD(\omega) \approx N_0^2 \frac{\Lambda^2 4\alpha^2\gamma}{(4\gamma^2 + \omega^2)} \frac{1}{(\omega^2\Lambda^2 + \rho^2)} = N_0^2 \cdot APSD_{\delta\rho}(\omega)\left|G_0(\omega)\right|^2. \tag{88}$$

Eq. (88) is identical with the case of power noise, Eq. (37).

4. CONCLUSIONS

It is thus seen that the unified theory is capable to reconstruct both the zero and the power reactor noise components in the corresponding limits. It also gives a correct description of the case when both types of noise are present concurrently. To our knowledge, this is the first case when the power reactor noise, i.e. the neutron noise induced by cross section fluctuations, was derived by master equations.

The model used in the present work was certainly very simple, but several restrictions can be alleviated and this is planned in the continuation of this work. Inclusion of delayed neutrons is relatively straightforward, at least conceptually. Considering more general forms of the cross section fluctuations is not so simple but also possible. Including space dependence, which is important for power reactor diagnostic problems, appears on the other hand prohibitively complicated with the master equation approach. For such cases the Langevin technique offers a much easier alternative to treat cases of practical interest.

One advantage of the present work is that it demonstrates that the treatment of power reactor noise is possible with a master equation approach, but it also demonstrates the difficulties associated with it. Thus it can be used as a reference for the justification of the Langevin approach for power reactor noise problems.

5. ACKNOWLEDGEMENTS

Part of this work was performed when one of the authors (I.P.) was visiting the Department of Chemical and Nuclear Engineering, University of New Mexico. Financial support for the visit from UNM is gratefully acknowledged.

6. REFERENCES

[1] R. P. Feynman, F. de Hoffman and R. Serber, *J. Nuclear Energy*, **3,** pp. 64(1956)

[2] J. A. Thie, *Reactor Noise,* Rowman & Littlefield, Inc. (1963)

[3] R. Uhrig, *Random Noise Techniques in Nuclear Reactor Systems*, The Ronald Press Company, New York (1970)

[4] M. M. R. Williams, *Random Processes in Nuclear Reactors*, Pergamon Press (1973)

[5] L. Pál, *Nuovo Cimento, Supplemento* **VII, 25** (1958); also, *Acta Phys. Hungarica* **XIV,** pp. 345-80 (1962)

[6] G. Bell, *Ann. Phys., NY* **21,** pp. 243 (1963)

[7] I. Pázsit and Y. Yamane, *Nucl. Sci. Eng.*, **133**, pp. 269 -281 (1999)

[8] M.M.R. Williams, Proc. R. Soc., Lond. *A* **358,** pp. 105 (1977)

[9] I. Pázsit, Note on the calculation of the variance in linear collision cascades. *J. Phys. D* **20,** pp. 151 - 161 (1987)

[10] I. Pázsit, *Physica Scripta*, **59**, pp. 344 - 351 (1999)

[11] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill Publishing Com., Second Edi.(1984)