SPECTRAL SOLUTION FOR TIME-DEPENDENT ONE-DIMENSIONAL TRANSPORT PROBLEM IN A SLAB

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ABSTRACT

In this work we solve the time dependent radiative problem combining the spectral and LTS_N methods. For such the angular flux $y(x,m,t)$ is expanded in the time variable, in a truncated Laguerre polynomial series :

$$y\,(x,m,t) = \sum_{k=0}^{L} L_k(t)\cdot y^{\,k}(x,m)$$

where $L_k(t)$ are the Laguerre polynomials. Replacing this in the transport equation and taking moments we come out with a set of steady-state problems which are solved by the LTS_N method. The final solution is read as

$$\Psi_N(x) = \left\{\sum_{k=1}^{M} e^{-s_k x} P_k\right\} A\Psi_N(0) + \sum_{k=1}^{M} e^{-s_k x} P_k * Q_N(x)$$

where P_k are the M matrizes of coefficients from the inversion of the Laplace transform, and s_k are the eigenvalues of the square matrix $s\mathbf{I} + \mathbf{A}$.

We are going to present the numerical results for values of M up 89.

We complete the mathematical analysis of this approach by proving the convergence employing tools of functional analysis.

1. INTRODUCTION

We consider a method that gives the efficient application of spectral method for solving the time-dependent one-dimensional transport problem. Spectral methods represent a fundamental approach to the numerical solution of partial differential equations and of integral-differential equations. A systematic analysis for the first type of equations was given by D. Gottlieb and S. Orszag [5], and a more comprehensive study was developed ten years later by Canuto, Hussaini, Quarteroni and Zang [4]. Spectral methods are very attractive for problems in several space dimensions in which high accuracy is required. In linear transport problems these methods are used by some authors involving different treatments for different variables [3,9,14]. In the approach of this work, we combine the hybrid LTS_N method [2,13], which employs the integral Laplace transform to the system of S_N-differential equations, and the spectral method for the time. Some natural extensions are possible for the time-dependent multi-dimensional transport problem, which is a straightforward task.

Consider the time-dependent one-dimensional transport problem (**1DTD**):

$$\frac{\partial \psi(x,t,\mu)}{\partial t} + \mu \frac{\partial \psi(x,t,\mu)}{\partial x} + \sigma_t \psi(x,t,\mu) = \int_{-1}^{1} k(x,\mu,\mu')\psi(x,t,\mu')d\mu' + q(x,t,\mu)$$

subject to the following conditions :

$$\begin{aligned} \psi(x,0,\mu) &= \varphi(x,\mu) \\ \psi(0,t,\mu) &= 0 \\ \psi(a,t,\mu) &= 0 \end{aligned}$$

where x represent the spatial variable ($0 < x < a$), t is the time ($t > 0$), μ is the angular cosine, ψ is the density flux function, q the source function, σ_t the total cross section and k is the scattering function. We give a treatment for the x variable and other for the t variable, and we begin choosing the discrete ordinates method. The steps of the hybrid method combining the LTS_N and the spectral methods are [8,14]:

- Apply the S_N method (discretizing the interval [-1,1] of angular cosines).
- Expand the angular flux in orthogonal polynomials in t .
- Apply the Laplace transform with respect to x to the system of S_N-differential equations.
- Solve for the transformed flux.
- Take the Laplace transform inverse to obtain the approximated flux.

The paper is organized as follows. In section 2, we review the hybrid LTS_N-spectral method for the time–dependent one-dimensional transport problem. In section 3, we give a survey of the convergence of the approximations generated by the method to the exact solution. In section 4, we analyze some numerical results derived of the method. Finally we arrive to some conclusions and outline the perspective of our work.

2. THE HYBRID LTS_N - SPECTRAL METHOD.

2.1. THE DISCRETE ORDINATES METHOD.

The discrete ordinates method establish a quadrature scheme in order to approximate the integral term subject to certain conditions. The basic facts of this method consists in selecting a finite number of discrete elements μ_m in [-1,1], and applying a quadrature scheme in the integral term of the transport equation. Remember that the summation

$$\sum_{m=1}^{M} \omega_m f(\mu_m) \quad converges \quad to \quad \int_{-1}^{1} f(\mu)\,d\mu$$

if the following requirements are accomplished [1] :

$$\begin{array}{lll} \mu_m \neq 0 \quad and \quad \omega_m \geq 0 & & \forall m = 1{:}\,M \\ \sup\limits_{m} \sum\limits_{m=1}^{M} \omega_m < \infty & & bounded \quad sum \quad of \quad weights \\ |\mu_l| = \min\limits_{1 \leq m \leq M} \{|\mu_m|\} & \Rightarrow & \sum\limits_{m=1}^{M} \dfrac{\omega_m}{|\mu_m|} \leq c\left(1 + \log|\mu_l|\right) \end{array}$$

for a universal constant c.
The resulting equations when we apply the discrete ordinates method is (**S_N-1DTD**) :

$$\frac{\partial \psi_m(x,t)}{\partial t} + \mu_m \frac{\partial \psi_m(x,t)}{\partial x} + \sigma_t \psi_m(x,t) = \sum_{n=1}^{M} \omega_n k_{nm} \psi_n(x,t) + q_m(x,t)$$

Subject to the following conditions :

$$\begin{aligned} \psi_m(x,0) &= \varphi_m(x) \\ \psi_m(0,t) &= 0 \\ \psi_m(a,t) &= 0 \end{aligned}$$

Where $\psi_m(x,t)$ represents the angular flux for the μ_m direction.

2.2. THE SPECTRAL METHOD.

Now we consider the expanded flux in a truncated Laguerre polynomials in t, *i.e.* [8,14] :

$$PROJ(\psi_m, t, L) = \sum_{l=0}^{L} \psi_m^l(x) L_l(t)$$

being L the order of series truncation, and $L_l(t)$ is the Laguerre polynomial of k-th degree. We denote this projection simply by $\psi_{m,L}$, but we continue denoting with the same symbol of the angular approximate flux ψ_m. The system of approximate equations of the system (**S_N-1DTD**) is :

$$\sum_{l=0}^{L} y_m^{\,l}(x)\frac{dL_l(t)}{dt} + m_m \sum_{l=0}^{L} \frac{dy_m^{\,l}(x)}{dx} L_l(t) + s_t \sum_{l=0}^{L} y_m^{\,l}(x) L_l(t) = \sum_{n=1}^{M} w_n k_{nm} \sum_{l=0}^{L} y_n^{\,l}(x) L_l(t) + q_m(x,t)$$

Taking moments we get

$$\sum_{l=0}^{L} g_j(l) y_m^{\,l}(x) + m_m (j!)^2 \frac{dy_m^{\,j}(x)}{dx} + (j!)^2 s_t y_m^{\,j}(x) = \sum_{n=1}^{M} w_n k_{nm} (j!)^2 y_n^{\,j}(x) + (j!)^2 q_m^j(x)$$

(m = 1,..., M and j = 0,..., L) where

$$g_j(l) = \int_0^{\infty} e^{-t} L_j(t) \frac{dL_l(t)}{dt} dt$$

$$q_m^j(x) = \int_0^{\infty} e^{-t} q_m(x,t) L_j(t) dt$$

2.3. APPLYING THE LAPLACE TRANSFORM AND SOLVING FOR THE FLUX.

No we apply the Laplace transform with respect to the spatial variable x obtaining :

$$\sum_{l=0}^{L} g_j(l) \hat{y}_m^{\,l}(s) + (j!)^2 m_m \left(s \hat{y}_m^{\,j}(s) - y_m^{\,j}(0)\right) + \left(j!\right)^2 s_t \hat{y}_m^{\,j}(s) = \sum_{n=1}^{M} w_n k_{nm} \left(j!\right)^2 \hat{y}_n^{\,j}(s) + \left(j!\right)^2 q_m^j(x)$$

where the hat represents the corresponding Laplace transform. In matrix notation we have :

$$(sA + B)\hat{\Psi}_N(s) = A\Psi_N(0) + \hat{Q}_N(s)$$

Next, we isolate the transformed flux by some inversion method, for example the recursive inversion method that combines the Schur and the partitioning methods. We find, according the work of Cardona (1996) the following solution :

$$\Psi_N(x) = \left\{ \sum_{k=1}^{L \cdot M} e^{r_k x} \Delta_k \right\} \cdot A\Psi_N(0) + \sum_{k=1}^{L \cdot M} e^{r_k x} \Delta_k * Q_N(x)$$

where Δ_k are the M matrices of coefficients from the inversion of the Laplace transform, ρ_k are the eigenvalues of the matrix $\mathbf{A}^{-1}\mathbf{B}$, and $Q_N(x)$ represents the source vector.

3. SURVEY OF THE CONVERGENCE OF THE LTS_N-SPECTRAL METHOD FOR THE TIME-DEPENDENT ONE-DIMENSIONAL TRANSPORT PROBLEM

Now our focus of attention is about the convergence of the LTS_N-spectral approximations. Keller gave the prove of the convergence of the S_N approximations for the steady-state one dimensional transport problem [6]. The approach of the C_0-semigroups was employed to prove the convergence of the LTS_N approximations for the one-velocity transport problem, both for the steady-state and time-dependent versions [7]. Some extensions were obtained to prove the convergence of the LTS_N-spectral approximations for the steady-state two-dimensional transport problem [8,9,14]. Now we give only a survey of the convergence of the LTS_N-spectral approximations for the time-dependent one-dimensional transport problem.

Let $\psi(x,t,\mu)$ be the exact flux, and let $\Psi_N(x,t)$ be the S_N-approximation vector of the flux and $\Psi_{N,L}(x,t)$ is the spectral- S_N-approximation vector of the flux, we define

$$\varepsilon_m(x,t) = \psi(x,t,\mu_m) - \psi_m(x,t)$$
$$\tau_m(x,t) = \int_{-1}^{1} \kappa(x,t,\mu_m,\mu')\psi(x,t,\mu')d\mu' - \sum_{n=1}^{M} \omega_n \kappa_{nm} \psi(x,t,\mu_n)$$

The first is the *error in the approximate flux*, and the last the *truncation error in the quadrature formula*. The following is a formula relating both errors (**S_N-errors**) :

$$\frac{\partial \varepsilon_m(x,t)}{\partial t} + \mu_m \cdot \frac{\partial \varepsilon_m(x,t)}{\partial x} + \sigma_t \varepsilon_m(x,t) = \sum_{n=1}^{M} \omega_n \kappa_{nm} \varepsilon_n(x,t) + \tau_m(x,t)$$

Now, we consider their approximations in truncated series of orthogonal polynomials of t :

$$\begin{cases} \varepsilon_{m,L}(x,t) = \displaystyle\sum_{l=0}^{L} \varepsilon_m^l(x) L_l(t) \\ \tau_{m,L}(x,t) = \displaystyle\sum_{l=0}^{L} \tau_m^l(x) L_l(t) \end{cases}$$

Next, we substitute the expressions of the expanded errors into the equation (**S_N-errors**), obtaining approximate equations and taking moments we get :

$$\mu_m \alpha_j \frac{d\varepsilon_m^l(x)}{dx} + \sum_{l=0}^{L} \gamma_j^l \varepsilon_m^l(x) + \alpha_j \sigma_t \varepsilon_m^j(x) = \sum_{n=1}^{M} \omega_n \kappa_{nm} \alpha_j \varepsilon_n^j(x) + \alpha_j \tau_m^j(x)$$

Where $\alpha_j = j\,!$ and γ_j^l is known. Multiply by $\varepsilon_m^j(x)$, sum with respect to *j* and integrate with respect to *x*, then

$$\frac{m_m}{2}\sum_{j=0}^{L}\left(\left(e_m^j(a)\right)^2-\left(e_m^j(0)\right)^2\right)+\sum_{j=0}^{L}\sum_{l=0}^{L}\frac{g_j^l}{a_j}\int_X e_m^j(x)e_m^l(x)dx+s_t\sum_{j=0}^{L}\int_X\left(e_m^j(x)\right)^2dx=$$

$$=\sum_{n=1}^{M}w_n k_{nm}\int_X\sum_{j=0}^{L}e_m^j(x)e_n^j(x)dx+\sum_{j=0}^{L}\int_X e_m^j(x)t_m^j(x)dx$$

In this and the following expressions of this section, we denote X = [-1,1]. Let us define the following functional vectors and the corresponding inner product :

$$e_m(x)=\left(e_m^j(x)\right)_{j=0:L}$$

$$t_m(x)=\left(t_m^j(x)\right)_{j=0:L}$$

$$\left\langle e_m(x),h_m(x)\right\rangle=\sum_{j=0}^{L}e_m^j(x)h_m^j(x)$$

We get, because the first term of the last equation is positive according the boundary conditions :

$$\int_X\left\langle e_m(x),Ne_m(x)\right\rangle dx+s_t\int_X\left\langle e_m(x),e_m(x)\right\rangle dx\le\sum_{n=1}^{M}w_n k_{nm}\int_X\left\langle e_m(x),e_n(x)\right\rangle dx+\int_X\left\langle e_m(x),t_m(x)\right\rangle dx$$

In this expression N is the matrix defined by

$$N=\left(\frac{g_j^l}{a_j}\right)_{\substack{j=0:L\\l=0:L}}$$

To this point we use the following notations :

$$e=\left(e_m\right)_{m=1:M}$$

$$\left\langle e\,\middle|\,h\right\rangle=\sum_{m=1}^{M}w_m\int_X\left\langle e_m(x),h_m(x)\right\rangle dx$$

$$\|e\|^2=\langle e\,|\,e\rangle$$

Let us multiply by ω_m and sum with respect to *m*, then

$$\langle e\,|\,Ne\rangle+s_t\langle e\,|\,e\rangle\le Ck_{max}\langle e\,|\,e\rangle+\langle e\,|\,t\rangle$$

Where $k_{\max}$ = *max*{k_{nm} | *n,, m* = 1:M }. We can rewrite this inequality as

$$\langle e\,|\,Ne\rangle+s_t\|e\|^2\le Ck_{max}\|e\|^2+C^{1/2}\|e\|\cdot\|t\|$$

In this case C is the sum of the weights of the quadrature, *i.e.*

$$C = \sum_{m=1}^{\infty} w_m$$

We use the following inequality

$$\|e\| \cdot \|t\| \leq \frac{1}{2}\left(d\|e\|^2 + \frac{\|t\|^2}{d} \right)$$

And now we obtain

$$\langle e \mid Ne \rangle + s_t \|e\|^2 \leq Ck_{max}\|e\|^2 + \frac{C^{1/2}}{2}\left(d\|e\|^2 + \frac{\|t\|^2}{d} \right)$$

This can be written as

$$\left\langle e \left| \left(s_t - Ck_{max} - \frac{C^{1/2}d}{2} + N \right) e \right\rangle \right. \leq \frac{C^{1/2}}{2d}\|t\|^2$$

Let us choose δ such that

$$Ck_{max} + \frac{C^{1/2}d}{2} = \frac{s_t}{2} \qquad or \qquad d = \frac{s_t}{\sqrt{C}} - 2\sqrt{C}k_{max}$$

In these circumstances we have the reduced inequality

$$\left\langle e \left| \left(\frac{s_t}{2} + N \right) e \right\rangle \right. \leq \frac{C}{2\left(s_t - 2Ck_{max} \right)}\|t\|^2$$

We define now

$$\tilde{e} = \sqrt{\frac{s_t}{2}}\left(I + \frac{2N}{s_t} \right)^{1/2} e$$

By the binomial series, if the matrix N is nilpotent, this expression is well defined and the above inequality becomes :

$$\|\tilde{e}\|^2 \leq \frac{C}{2\left(s_t - 2Ck_{max} \right)}\|t\|^2$$

This expression says that the error in the approximate flux is lesser than a constant times the truncation error in the quadrature formula. For the Gauss-Legendre quadrature, C = 2 , and the inequality takes the form :

$$\|\tilde{\varepsilon}\| \le \frac{1}{\sqrt{\sigma_t - 4\kappa_{max}}}\|\tau\|$$

In fact, the matrix N is nilpotent and its order of nilpotency is L, the order of truncation in expanded series of the Laguerre polynomials.

4. NUMERICAL RESULTS.

We consider the test problem in which radiation penetrates into a purely absorbing, initially cold, slab. The radiation is driven by a time-independent, isotropic intensity incident upon the left side of the slab for all times t > 0, with a vacuum condition on the right side of the slab, such as is in Szilard and Pomraning [12]. The governing equation is

$$\frac{1}{C}\frac{\partial I}{\partial t}(x,\mu,t) + \mu\frac{\partial I}{\partial x}(x,\mu,t) + \sigma_t I(x,\mu,t) = \frac{1}{2}\int_{-1}^{1}\sigma_s(\mu',\mu)I(x,\mu',t)d\mu'$$

where $I(x,\mu,t)$ is the radiative intensity, t represents the time, c the velocity of light, σ_t is the total cross section, and with the following boundary and initial conditions :

$$I(0,\mu,t) = \Gamma_1(\mu,t) \qquad for \quad \mu > 0$$

$$I(a,\mu,t) = \Gamma_2(\mu,t) \qquad for \quad \mu < 0$$

$$I(x,\mu,0) = I_0(x,\mu)$$

The situation considered above correspond to the following parameters :

L = 0,1	c = 1	$\sigma_t = 1$	$\sigma_s = 1$	$I(0,\mu,t) = 2$ for $\mu > 0$	$I(0,\mu,t) = 0$ for $\mu < 0$	$I(x,\mu,0) = 0,5\ (10^{-10})$

The radiation temperature is :

$$T_r^4(x,t) = \int_{-1}^{1} I(x,\mu',t)d\mu'$$

The spectral solution for the problem is known. The following are the numerical results obtained by Renz [11] for $N = 2$, and $L = 85, 86, 87$.

Table I. Example Table.

X	T_r^4 L=85	T_r^4 L=86	T_r^4 L=87
0,00	1,07744	1,07753	1,07759
0,01	1,07293	1,07375	1,07444
0,02	1,06946	1,06991	1,07023
0,03	1,06637	1,06618	1,06592
0,04	1,06327	1,06258	1,06189
0,05	1,05992	1,05903	1,05819
0,06	1,05624	1,05544	1,05472
0,07	1,05223	1,05172	1,05131
0,08	1,04793	1,04782	1,04781
0,09	1,04341	1,04373	1,04412
0,10	1,03874	1,03944	1,04017

Plotting the values of the third column of the Table I with respect to *x*, we obtain :

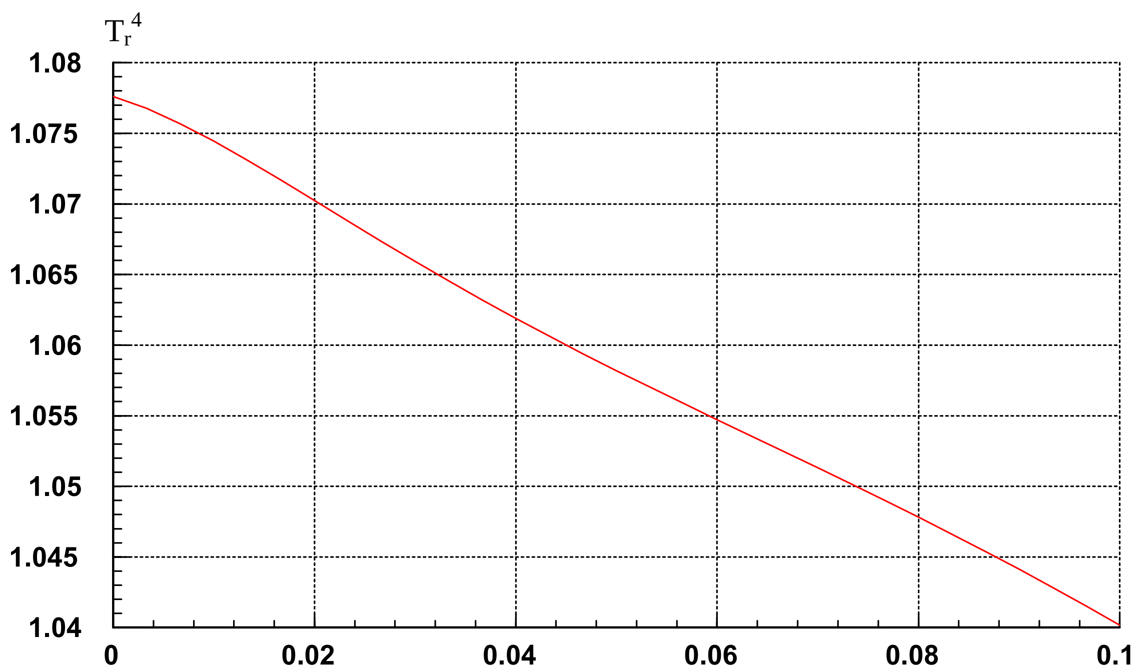

Figure 1. Radiative temperature for t = 5 and *L* = 87.

For all other values of *L*, the results are coincident; this means the numerical convergence of the searched solution. The same occurs when we consider different values of *N*. These results were obtained both in a Fortran 90 program. For the matrix inversion it was employed the recursive inversion method that combines the Schur and partitioning methods. The convolution was obtained in a elementary way with the trapeze rule.

CONCLUSIONS

In this paper was considered an hybrid method to solve the time-dependent one-dimensional transport problem, the LTS_N-spectral method, a different treatment for each variable. Then, with some tools of functional analysis was given a survey of the convergence of the LTS_N-spectral approximations; the cornerstone is the convergence of the steady-state LTS_N approximations [7], because in our approach with the spectral method the original problem is approximated by schemes depending only of the spatial variable. It was established an important relation between The error of the approximative flux and the error of the truncation in the quadrature formula.

Many aspects such as the convergence velocity, and the study of the case when the total cross section or the scattering kernel hinge on the spatial variable are the next steps in our work. The analysis of the hybrid method is not complete at all, an excellent challenge for the future.

Some extensions are possible, by considering the time-dependent two- or multi- dimensional transport problems, in which case we could obtain a comparative frame with finite-difference and finite-element methods, is it a straightforward task.

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