COARSE MESH FINITE DIFFERENCE METHODS AND APPLICATIONS

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ABSTRACT

The more rigorously formulated Coarse Mesh Finite Difference (CMFD) method and the Analytic CMFD method recently proposed by the author are compared in-depth in a unified treatment with various other CMFD formulations from the perspectives of both theory and computation. Implementation and applications of both the proposed CMFD and ACMFD methods in a core design code are discussed, and numerical results presented. The CMFD formulation is also generalized for transport calculation application, resulting in a form identical to that for the diffusion case except for a different definition of the diffusion coefficient. The appropriate modification of the CMFD formulation for problems with fixed source is worked out as well.

1. INTRODUCTION

Conventionally a Coarse Mesh Finite difference (CMFD) method uses empirically determined coupling coefficients to set up a very simple nodal balance equation involving only node average fluxes as the unknowns. It has become a popular practice to apply iterative CMFD to advanced nodal methods to speed up calculation (Ref.1 to 3). CMFD corrective coefficients are introduced and iteratively updated using the evolving advanced nodal method solution over two adjacent nodes. The corrective coefficients are heuristic, and are determined to reproduce nodal surface currents (Ref.1 and 2) and nodal surface fluxes (Ref.3).The higher level method accelerated by CMFD is not restricted to advanced nodal methods. In Ref.4, a CMFD method, similar to that of Ref.3 but introduced much earlier, was used to accelerate higher level fine mesh finite difference calculation.

Recently the author presented a more rigorous formulation of CMFD (Ref.5). It was shown how directional dependent Effective Diffusion Coefficient (EDC) can be unambiguously defined and rigorously derived for the general case to relate in finite difference forms, both node surface currents and node surface fluxes as well, to node average fluxes. These relations can be used to

cast any advanced nodal method to the equivalent CMFD form to gain acceleration, by performing iteratively the CMFD and the advanced nodal calculations.

It was also shown in Ref.5 that for the case of homogeneous nodes, a self-contained and stand-alone Analytic Coarse Mesh Finite Difference (ACMFD) method results. For ACMFD, closed form analytic expressions for EDCs were explicitly derived in both Cartesian and hexagonal geometries. The simplicity of ACMFD provide a variety of attractive potential applications such as loading pattern search.

Reference 5 focused on the derivation of the theoretical formulation for CMFD and ACMFD in Cartesian and hexagonal geometries. This paper discusses the implementation and applications of the results of Ref.5, and presents a generalization of the CMFD formulation to transport calculation and fixed source calculation. Section 2 presents a unified treatment of the various CMFD formulations and provides an in-depth comparison of the different formulations from the perspectives of both theory and computation. Section 3 discusses the implementation of CMFD and ACMFD in the Westinghouse core design code ANC. Section 4 presents the generalization of the CMFD method to transport calculation and fixed source calculation. Section 5 concludes the paper.

2. COMPARSION OF CMFD FORMULATIONS

2.1 FROM THE PERSPECTIVE OF THEORY

It is useful to review the formulations proposed in Ref.1 (Smith formulation), Ref.3 (referred as the MCNH formulation), Ref.4 (referred as the AA formulation) and Ref.5 (Chao formulation) from a perspective that will put all of them in the same scheme to help comparison. Consider two adjacent homogeneous nodes of unity pitch distance as shown in Fig.1.

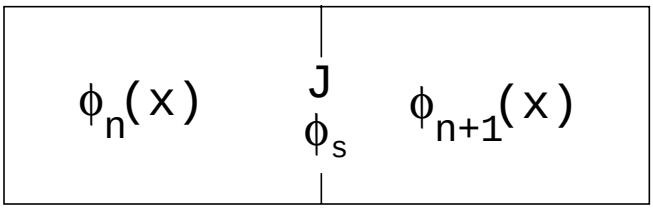

Figure 1

The classical fine mesh finite difference equation is derived by assuming that the interface current J, the interface flux ϕ_s, and the node average fluxes, $\bar{\phi}_n$ and $\bar{\phi}_{n+1}$, are related as,

$$J = -2D_n(\phi_s - \bar{\Phi}_n) \quad (1a)$$

$$J = 2D_{n+1}(\phi_s - \bar{\Phi}_{n+1}) \quad (1b)$$

Eliminating the interface flux or current gives

$$J = -D_{fm}(\bar{\Phi}_{n+1} - \bar{\Phi}_n) \tag{2a}$$

$$\phi_s = \frac{1}{2D_n} D_{fm}\bar{\Phi}_{n+1} + \frac{1}{2D_{n+1}} D_{fm}\bar{\Phi}_n \tag{2b}$$

where D_{fm} is the conventional fine mesh diffusion coefficient which equals the harmonic mean of the diffusion coefficients of the two adjacent nodes,

$$D_{fm} = 2\left[\frac{1}{D_n} + \frac{1}{D_{n+1}}\right]^{-1} \tag{3}$$

The Smith formulation assumes that the interface current J and the node average fluxes, $\bar{\phi}_n$ and $\bar{\phi}_{n+1}$, from the higher level nodal method solution satisfy the following finite difference relation,

$$J = -D_{fm}[(\bar{\Phi}_{n+1} - \bar{\Phi}_n) + C(\bar{\Phi}_{n+1} + \bar{\Phi}_n)] \tag{4}$$

Comparing with eq.(2a), we see that here the first term is in the classical fine mesh finite difference form and the second term is a coarse mesh correction term with the correction factor *C* to be determined such that eq.(4) is satisfied. It will be illuminating to point out that the Smith formulation of eq.(4) can be derived by modifying the classical fine mesh finite difference relations, eqs.(1), by adding the coarse mesh correction term as,

$$J = -2D_n(\phi_s - \bar{\Phi}_n + C\bar{\Phi}_n) \tag{5a}$$

$$J = 2D_{n+1}(\phi_s - \bar{\Phi}_{n+1} - C\bar{\Phi}_{n+1}) \tag{5b}$$

However, since there is only one unknown correction factor *C*, eqs.(5a) and (5b) can not be both satisfied by the higher level nodal method solution. Thus only eq.(4) is usable in the Smith formulation.

The MCNH formulation modifies the classical fine mesh finite difference relations, eqs.(1), in a way different from Smith's method by introducing two instead of one coarse mesh correction factors as follows,

$$J = -2D_n(\phi_s - \bar{\Phi}_n) - C_n^R(\phi_s + \bar{\Phi}_n) \tag{6a}$$

$$J = 2D_{n+1}(\phi_s - \bar{\Phi}_{n+1}) + C_{n+1}^L(\phi_s + \bar{\Phi}_{n+1}) \tag{6b}$$

which should be compared against eqs.(5). With two unknowns, both eqs.(6a) and (6b) can be required to be satisfied by the higher level nodal method solution so that the directional dependent correction factors C^R and C^L can be determined. From these two equations, J and ϕ_s can each be expressed in finite difference forms in terms of the node average fluxes as,

$$J = -D_{n+1}^{L}\bar{\Phi}_{n+1} + D_n^R\bar{\Phi}_n \tag{7a}$$

$$\phi_s = A_1\bar{\Phi}_{n+1} + A_2\bar{\Phi}_{n+1} \tag{7b}$$

where the coefficients are somewhat involved functions of the parameters D, C^R and C^L of eqs.(6).

The AA formulation is similar to the MCNH formulation, although introduced much earlier for accelerating fine mesh finite difference calculations. Instead of eqs.(6), the AA formulation modifies eqs.(1) in a similar but different way as follows,

$$J = -2D_n(C_n^R\phi_s - \bar{\Phi}_n) \tag{8a}$$

$$J = 2D_{n+1}(C_{n+1}^L\phi_s - \bar{\Phi}_{n+1}) \tag{8b}$$

and the rest is the same as in the MCNH formulation.

All the above three CMFD formulations are different ways of introducing coarse mesh correction factors to the classical fine mesh finite difference eqs.(1). The correction factors are introduced in an ad hoc manner without apparent theoretical basis. In Ref.5, Chao looked into the question if the coarse mesh generalization of eqs.(1) can be rigorously derived. It was straightforward to show that for the above one dimensional problem, the following rigorous relations result,

$$J = -2\frac{D_n}{C_n^j}\left(\phi_s - C_n^f\bar{\Phi}_n\right) \tag{9a}$$

$$J = 2\frac{D_{n+1}}{C_{n+1}^j}(\phi_s - C_{n+1}^f\bar{\Phi}_{n+1}) \tag{9b}$$

where the coarse mesh correction factors are explicitly given as

$$C^j = \frac{\tan(k/2)}{k/2} \tag{10a}$$

$$C^f = \frac{k}{\sin k} \qquad (10b)$$

where

$k = (k_\infty - 1)^{1/2}$ pitch / (diffusion length) (10c)

Comparing eqs(9) to eqs.(5), (6) and (8), we see that none of the three proposed formulations has the correct form. A closer study reveals that the MCNH formulation, eqs.(6), is the closest among the three to the Chao formulation, eqs.(9). Actually eqs.(6) is the first order approximation to eqs.(9). In the limit of k approaching zero, both C^j and C^f approach one. In this limit, the zeroth order approximation to eqs.(9) reproduces eqs.(1), and the first order approximation to eqs.(9) gives eqs.(6). The zeroth order approximation is easy to see. The first order approximation takes a little work, arriving at the result that when k is small eqs.(9) reduces to eqs.(6) with the MCNH coarse mesh correction factor C in eqs.(6) equal to $-(k^2D/6)$. This small k limit happens according to eq.(10c) either as k-infinity approaches one or as the pitch distance is small compared to the diffusion length, which is not true for the coarse mesh in nodal methods nor for nodes with k-infinity far from being critical such as nodes with control rods.

Eliminating the interface flux or current in eqs.(9), the Chao formulation provides the following relations,

$$J = -\left[D^L_{n+1}\ \bar{\Phi}_{n+1}\ - D^R_n \bar{\Phi}_n\right] \qquad (11a)$$

$$\phi_s = C^j_n \frac{1}{2D_n} D^L_{n+1}\ \bar{\Phi}_{n+1} + C^j_{n+1} \frac{1}{2D_{n+1}} D^R_n \bar{\Phi}_n \qquad (11b)$$

where the directional dependent EDCs are given as

$$D^L_n = 2\left[C^j_n \frac{1}{D_n} + C^j_{n-1} \frac{1}{D_{n-1}}\right]^{-1} C^f_n \qquad (12a)$$

$$D^R_n = 2\left[C^j_n \frac{1}{D_n} + C^j_{n+1} \frac{1}{D_{n+1}}\right]^{-1} C^f_n \qquad (12b)$$

The striking similarity of eqs.(11) and (12) to the classical fine mesh finite difference eqs.(2) and (3) is worth noting. Although these results are derived for the simple one group one dimensional case, it was shown in Ref.5 that they can be analytically generalized to two group multi-dimensional homogeneous node cases for either A) Cartesian geometry with polynomial transverse leakage profiles constructed over adjacent nodes, or B) hexagonal geometry using a radial expansion with six plane wave eigen-functions perpendicular to the six radial surfaces. Thus for cases of homogeneous nodes, in both Cartesian and hexagonal geometries, Analytic

CMFD (ACMFD) formulation was explicitly derived in Ref.5, which is self-contained and can be used standalone without the need of any higher level nodal method calculation.

For the general case application regardless of what kind of problem and which higher level method to be accelerated, eqs.(11) provide the basis of generalization. Instead of using the EDCs explicitly given by eqs.(12), one can determine the two EDCs in eqs.(11) by requiring that the higher level method solution satisfies eqs.(11), which leads to the following general definition of the EDCs in the Chao formulation of CMFD,

$$D_n^L = 2\left[C_n^j \frac{1}{D_n} + C_{n-1}^j \frac{1}{D_{n-1}}\right]^{-1}\left[\frac{\Phi_n^L}{\bar{\Phi}_n} - C_n^j \frac{J_n^L}{2D_n\bar{\Phi}_n}\right] \qquad (13a)$$

$$D_n^R = 2\left[C_n^j \frac{1}{D_n} + C_{n+1}^j \frac{1}{D_{n+1}}\right]^{-1}\left[\frac{\Phi_n^R}{\bar{\Phi}_n} + C_n^j \frac{J_n^R}{2D_n\bar{\Phi}_n}\right] \qquad (13b)$$

where Φ^L (Φ^R) and J^L (J^R) refer to the surface averaged flux and surface averaged current on the left (right) surface of a node. For the special case of one dimensional homogeneous nodes, eqs.(13) reduce to eqs.(12).

2.2 FROM THE PERSPECTIVE OF COMPUTATION

For generic applications as acceleration methods, all the above four CMFD formulations use the higher level method solution to determine the CMFD coefficients. If the exact higher level solution is available or converged to, any one of the four formulations can provide a CMFD matrix equation completely satisfied by the true solution. Thus the question arises how different the various formulations are in practical applications, that is, from the perspective of numerical calculation. To answer this question it will be useful to revisit the four formulations presented in the above from a different angle.

The goal of any CMFD formulation is to obtain a relation for J linear in $\bar{\phi}_n$ and $\bar{\phi}_{n+1}$, like eq.(11a), which for the convenience of discussion here will be called the CMFD current relation. The two coefficients in this relation can be interpreted as directional dependent Effective Diffusion Coefficient (EDC). Once the EDCs are obtained, plugging the CMFD current relation into the neutron balance relation immediately gives the CMFD matrix equation for the eigen-value problem to be solved. So the basic question of CMFD formulation is how to determine the two EDCs. In general two conditions are required. Since the two EDCs are going to be determined using the (evolving) higher level method solution, the CMFD current relation itself can be counted as one of the two required conditions. Hence one more condition is needed to complete a CMFD formulation. This extra degree of freedom is the root cause of ambiguity in CMFD formulation. In principle, for a given higher level method solution, one could make any combinations of the two EDCs to construct an eigen-value matrix equation satisfied by the reference solution. However, obviously not all these EDC choices will lead to acceptable CMFD.

For example, the simplest choice is to set the two EDCs the same and let the CMFD current relation itself determine "the EDC". By doing so one could easily run into negative EDC and lose the positive definiteness of the CMFD matrix equation. For instance, the interface current can be flowing from a a lower average flux node to a higher average flux node. This is a very simple but very important point to realize. Although given the exact solution of the problem, one can always construct a certain CMFD current relation for the solution to satisfy, this by no means says that the resulting CMFD matrix equation itself has all the necessary mathematical features to be used for iteratively solving the eigen-value problem. Furthermore, during the iteration process the exact solution is not known, the evolving CMFD matrix equation must be iteratively solvable throughout the process and lead to the correct answer at the end. Therefore two conditions must be met by a CMFD method. One is that throughout the iteration process the evolving CMFD must have the required matrix properties such that the largest modular eigen-value is real and positive, and the corresponding eigen-function is the only real and positive solution. Two is that the evolving CMFD matrix equation must bear sufficient fidelity to the higher level method equation in order to stably and effectively converge to the higher level method solution. The first condition can be mathematically analyzed due to the well established properties of the finite difference diffusion equation matrix. This is an interesting problem for investigation, but will not be done here. The second condition seems difficult to analyze mathematically, because of the non-linear feature of the evolution. However, it is clear that both conditions will be better met if the EDCs are so chosen that the resulting CMFD matrix is closer to being a dominant of the original matrix equation for the higher level method.

The Smith formulation, eq.(4), can be obtained from the CMFD current relation eq.(11a) with the second condition that the average of the two EDCs must equal to D_{fm}, i.e. the conventional fine mesh finite difference diffusion coefficient. In either the MCNH or the AA formulation, eq.(7a) can be rewritten as the CMFD current relation eq.(11a) with the second condition as eq.(7b), where the coefficients A_1 and A_2 are expressible in terms of the two EDCs. But these expressions are somewhat complicated and not transparent, as one has to first work out the dependency of A_1, A_2, and the two EDCs, on the parameters C^R and C^L via eqs.(6) or eqs.(8), and then eliminate C^R and C^L in order to express A_1 and A_2 in terms of the two EDCs. In the Chao formulation, the CMFD current relation is eq.(11a) and the second condition is eq.(11b). The fact that all the four methods can be viewed as using the same CMFD current relation, but using a different second condition to determine the two EDCs, means that even if the same reference solution is given, the four methods will still have different CMFD matrices. Since the four different matrix equations, by construction, are all satisfied by the same reference solution, the reference solution will be the fundamental mode of all the four matrix equations if the required matrix conditions are satisfied by all the four matrices. But all the other excited modes will be different for the four different matrix equations. Although iterations on the four different matrix equations will all lead to the same reference solution, the efficiency of the iteration will be different. Furthermore, in practice the reference solution is not known and the CMFD matrix is constructed from an evolving higher level method solution which never satisfies the CMFD matrix equation as its eigen-function solution until the end of the evolution process. In fact it is exactly the difference of the eigen-function of the evolving CMFD equation from the evolving higher level method solution that is accelerating the process. But if this difference is not moving in the direction toward the correct answer, instability arises.

The fact that the Chao formulation is rigorous for the one-dimensional case and furthermore was shown in Ref.5 to be analytically generalizable to multi-dimensional cases, is a strong argument that this formulation meets, or better meets the above discussed two required conditions for CMFD.

3. APPLICATIONS IN DIFFUSION CODES

3.1 CMFD ACCELERATING DIFFUSION METHODS

Here we will focus mainly on the discussion of using CMFD to accelerate a higher level nodal diffusion method, although CMFD can be applied equally well to accelerate the fine mesh finite difference method as was done in Ref.3. In design code applications, it is necessary to consider non-homogeneous nodes to account for either depletion burnup gradient or spectrum interaction between adjacent nodes. Lacking the capability of handling possible node in-homogeneity, the closed form Analytic CMFD (ACMFD) can not be implemented in design code for generic applications. For generic design applications, CMFD needs to be used iteratively with the higher level nodal method as an acceleration tool. This has been done to the Westinghouse advanced nodal diffusion code, ANC. ANC uses a Nodal Expansion Method (NEM), which contains polynomial and exponential terms in the expansion. The NEM formulation also applies to the hexagonal geometry via the technique of using conformal mapping to convert the hexagonal geometry to Cartesian geometry.

Unlike the approach in Refs.1 to 4, where a two or few nodes higher level method calculation is performed to iterate with a whole core CMFD calculation, the CMFD option in ANC is directly coupled to the original whole core NEM calculation. In the original whole core NEM calculation, outer (source) iterations are performed for every cross-section feedback iteration. When the CMFD option is used, for every feedback iteration, only one NEM source iteration is performed. Using the eigen-value, the interface fluxes and currents, and the node average fluxes from the whole core NEM solution, the directional dependent EDCs for CMFD in all the nodes are calculated simultaneously according to eqs.(13). The equivalent finite difference equation of CMFD is used for a lower level stand alone outer/inner iterative calculations including source extrapolation acceleration. The flux solution so obtained is used to update the neutron source in the higher level whole core NEM calculation at the next cross-section feedback iteration. This CMFD acceleration option in ANC works very effectively and its performance is very robust for both Cartesian and hexagonal geometry applications. The CMFD option has therefore been made the default option of ANC.

The advantage of implementing CMFD the way described above are two folds. One is that the EDCs determined are more consistent with the latest NEM solution as compared to using only a two or few nodes NEM calculation. The other is that the original whole core NEM calculation option is kept intact, so that at any time one can revert back to the original whole core NEM calculation option if necessary. Although in practice we have not had any stability or convergence difficulty in using our CMFD method, however as discussed in Section 2.2, unlike using ACMFD there is a lack of theoretical proof that the non-linear iteration between NEM and

CMFD would never lead to a pathological CMFD matrix that could cause a convergence or stability problem. Should any such case arise, this direct coupling of whole core NEM and CMFD calculations could allow one to bypass the CMFD calculation and could also provide a convenient interface to investigate the problem. It is worth mentioning in this regard that in the early stage of developing our CMFD method, we tried to use the Smith formulation as a test. In the test cases we tried, we did experience convergence and stability problem, which nevertheless could be alleviated by performing more NEM outer iterations before going to the CMFD calculation at each NEM and CMFD interface iteration.

3.2 STANDALONE ACMFD

As mentioned in Section 2.2 following eqs.(12), the rigorous CMFD formulation for the one group one dimensional case has been analytically generalized in Ref. 5 to two group multi-dimensional homogeneous node cases. Explicit equations were provided in Ref.5 for the case of Cartesian geometry with flat surface current profile and the case of hexagonal geometry with a radial flux expansion in six plane wave eigen-functions. In this ACMFD method, the unknowns in the analytic expressions for the EDCs are the eigen-value and the ratio of the transverse leakage to the node average flux, all other constants being expressible in terms of the node group constants. Therefore ACMFD does not need a higher level nodal code spatial solution in order to determine the EDCs. The EDCs can be upgraded analytically on a node by node basis using the most recent solution of ACMFD itself. The flat current profile in the Cartesian geometry case is not a necessary restriction and can be generalized to a low order polynomial straightforwardly. However, in practice the flat profile assumption in the Cartesian node case is perhaps comparable or no worse than the homogeneous node assumption. Therefore in our current implementation of ACMFD we make the simple flat profile assumption for the Cartesian geometry case.

ACMFD has been implemented in ANC as an additional option. The ACMFD option in ANC bypasses the original NEM calculation. At every feedback iteration in ANC, the EDCs of ACMFD are upgraded analytically once. The subsequent outer and inner iterations are then performed with the same module that the CMFD option uses. The major assumption of ACMFD is the node homogeneity. In modeling commercial cores in Cartesian geometry, four nodes per assembly are usually used, in which case the assumption of node homogeneity is quite good. However, for hexagonal geometry core modeling, a hexagonal node can not be sub-divided into smaller hexagonal nodes. In this case, it is still possible to partially account for the burnup gradient effect in ACMFD, even though in a less accurate way as compared to NEM. Spatial flux distribution can still be calculated from the ACMFD solution which contains both node average and node surface fluxes. The node cross-section spatial distribution is weighted with the spatial flux distribution to obtain the node effective cross-section in order to partially account for the burnup gradient effect.

The ACMFD option of ANC is qualified with both numerical benchmark problems and commercial core design models. The accuracy of ACMFD is found to be very good, quite acceptable for many interesting applications. The simplicity of ACMFD is very attractive for a variety of applications, in particular for loading pattern search applications where a variety of

types of calculations at different levels of accuracy are needed at different stages of the search process. The ACMFD matrix is easily adaptable to a variety of approximations. It is also easy to use for perturbation calculations or linearization approximations.

3.3 NUMERICAL RESULTS

As an example to demonstrate the performance of the CMFD and ACMFD methods in ANC, results of a full core stuck rod calculation are presented in the following. The core considered is a Westinghouse type 3-loop PWR reload core. The initial state is at the all rods out full power condition near the end of a reload cycle. The core is then brought down to hot zero power condition with all rods inserted except for one peripheral rod being stuck. The stuck rod is on a symmetry axis that the core power distribution is top/bottom symmetric, although full core calculation is performed. The radial power distribution is greatly skewed and the axial power distribution axial offset changed from the initial -2% to 26%. This example is chosen to show a more severe test case.

Table 1 compares the relative running time of using NEM/CMFD or ACMFD versus using NEM only. Compared to using NEM only, the iterative flux solution times of using NEM/CMFD and ACMFD are respectively 50% and 30%. For the total running time comparison, the corresponding ratios are 60% and 40% respectively. The total time includes pin power reconstruction, cross-section processing and input/output in addition to iterative flux solution. The flux solution time includes all calculations in the inner/outer/feedback iteration process, including burnup gradient, spectrum interaction and boron search, etc.

Table 1: Relative Running Time Comparison For the Full Core Stuck Rod Problem

	NEM only	NEM/CMFD	ACMFD
Iterative Flux Solution	1.0	0.49	0.30
Total Run	1.22	0.71	0.48

The solution of using NEM/CMFD is essentially identical to that of using NEM only, as it should be. It is interesting to see how good the ACMFD solution is as compared to the reference solution, which is taken to be the one of using NEM/CMFD. Table 2 compares the predictions of the major global core parameters.

Table 2: Comparison of Predictions For Major Global Core Parameters

	Boron Concen-tration (ppm)	Peak Local Power	Peak Radial Pin Power	Axial Off-set (%)
NEM/CMFD	-507.5	5.543	3.779	25.5
ACMFD	-501.1	5.561	3.776	26.1

The boron concentration is off by 6.4 ppm, but the peak local power, peak radial pin power, and axial off-set are all in excellent agreement. Figure 2 shows a more detailed radial power

distribution comparison, where the average power in assemblies of the top half of the core is displayed. The comparison can be seen to be excellent as well. We see that even for this severe test case, the accuracy of ACMFD is more than being adequate for many applications. It can be further approximated for various types of applications in loading pattern search.

						0.275	0.351	0.269						
						0.277	0.357	0.271						
				0.337	0.333	0.746	0.500	0.722	0.318	0.379				
				0.343	0.336	0.759	0.497	0.733	0.320	0.384				
			0.488	1.050	0.902	0.716	1.114	0.688	0.837	0.937	0.401			
			0.498	1.070	0.914	0.711	1.115	0.682	0.845	0.948	0.405			
		0.635	0.630	0.939	0.782	1.374	2.000	1.318	0.704	0.772	0.444	0.378		
		0.647	0.636	0.948	0.778	1.371	2.005	1.313	0.697	0.773	0.444	0.381		
	0.701	1.591	1.186	0.765	1.130	1.900	1.645	1.791	0.979	0.569	0.742	0.861	0.282	
	0.712	1.621	1.200	0.754	1.129	1.900	1.636	1.787	0.973	0.555	0.742	0.868	0.283	
	0.777	1.604	1.136	1.385	0.916	1.130	0.835	1.027	0.724	0.949	0.669	0.778	0.291	
	0.776	1.627	1.132	1.383	0.911	1.128	0.823	1.022	0.716	0.942	0.661	0.782	0.290	
1.165	2.625	1.588	2.167	2.587	1.354	0.728	0.832	0.627	0.990	1.703	1.245	0.649	0.677	0.251
1.172	2.645	1.577	2.159	2.582	1.350	0.714	0.824	0.613	0.983	1.694	1.235	0.640	0.683	0.251
1.786	3.458	2.752	3.303	2.364	1.081	0.925	0.919	0.789	0.773	1.527	1.859	1.035	0.464	0.325
1.798	3.467	2.747	3.295	2.346	1.066	0.917	0.905	0.779	0.760	1.512	1.855	1.030	0.457	0.328

NEM/CMFD
ACMFD

Figure 2. Comparison of Assembly Power Distributions of ACMFD Versus NEM/CMFD

4. GENERALIZATION TO TRANSPORT AND FIXED SOURCE CALCULATIONS

In core analysis type transport calculations, just as in diffusion calculations, the neutron balance relation provides the major matrix equation for eigen-value problem iterations. Although the angular fluxes are calculated in a much more complicated way, at the end of each iteration the average currents on the surfaces of a (small) homogeneous node are calculated and plugged into the same neutron balance equation as in the case of diffusion calculations. At this stage of using the neutron balance relation, the node average flux, the node surface fluxes and surface currents are all available. Therefore one should be able to employ the same idea of using CMFD to accelerate the transport calculation. Eqs.(13) can still be used to calculate the EDCs for a transport CMFD calculation. However, the known coefficients in eqs.(13), i.e. D and C^j, need to be generalized from the diffusion case to the transport case.

To derive the equivalent of eqs.(13) for the transport case, we will follow the approach in Ref.(5) to first derive the equivalent of eqs.(9) for the one group one dimensional case. Using standard notations, the one dimensional transport equation in a homogeneous node with isotropic scattering (assuming the conventional angular correction included) is

$$\mu\frac{\partial}{\partial x}\psi(x,\mu)+\Sigma_t\psi(x,\mu) = \frac{\Sigma_t c}{2}\phi(x)+\frac{S}{2} \tag{14}$$

The source in the (small) node is always assumed to be flat, and therefore can be easily accounted for with the constant particular solution of $S/(2(1\text{-}c)\Sigma_t)$. However, as we will see, the presence of fixed sources requires a generalization of the CMFD current relation, eq.(11a). We will discuss the two cases of without and with fixed source separately in sections 4.1 and 4.2.

4.1 CASE WITHOUT FIXED SOURCE

Following Ref.6, the general solution to eq.(14) is written as an expansion in eigen-functions,

$$\psi(x,\mu) = A_+ f(\upsilon_o,\mu)e^{-\Sigma_t x/\upsilon_o} + A_- f(-\upsilon_o,\mu)e^{\Sigma_t x/\upsilon_o} + \int_{-1}^{1} A(\upsilon) f(\upsilon,\mu)e^{-\Sigma_t x/\upsilon}\,d\upsilon \tag{15}$$

where the first two terms are the discrete asymptotic modes and the integral is over the continuous transient modes. The eigen-value υ_o always has a magnitude greater than one and satisfies the following equation,

$$c\upsilon_o\tanh^{-1}(1/\upsilon_o) = 1 \tag{16}$$

The eigen-functions satisfy the following conditions,

$$\int_{-1}^{1} f(\upsilon,\mu)d\mu = 1 \tag{17a}$$

$$\int_{-1}^{1} f(\upsilon,\mu)\mu d\mu = (1-c)\upsilon \tag{17b}$$

Using the conditions in eqs.(17) we find through the appropriate integrations over eq.(15) that the scaler flux $\phi(x)$, the net current $J(x)$, and the average flux $\bar{\phi}$ over the node of a unity width are respectively given by,

$$\phi(x) = (A_+ + A_-)\cosh\frac{\Sigma_t x}{\upsilon_o} - (A_+ - A_-)\sinh\frac{\Sigma_t x}{\upsilon_o} + \int_{-1}^{1} A(\upsilon)e^{-\Sigma_t x/\upsilon}\,d\upsilon \tag{18}$$

$$\frac{J(x)}{(1-c)\upsilon_o} = -(A_+ + A_-)\sinh\frac{\Sigma_t x}{\upsilon_o} + (A_+ - A_-)\cosh\frac{\Sigma_t x}{\upsilon_o} + \int_{-1}^{1} A(\upsilon)\frac{\upsilon}{\upsilon_o}e^{-\Sigma_t x/\upsilon}d\upsilon \quad (19)$$

$$\bar{\phi} = (A_+ + A_-)\frac{2\upsilon_o}{\Sigma_t}\sinh\frac{\Sigma_t}{2\upsilon_o} + \int_{-1}^{1} A(\upsilon)\frac{2\upsilon}{\Sigma_t}\sinh\frac{\Sigma_t}{2\upsilon}d\upsilon \quad (20)$$

Combining eqs.(18) and (19) to eliminate (A_+-A_-) and then substituting (A_++A_-) via eq.(20), the following relation among the scaler flux, the net current and the average flux is obtained,

$$\phi(x) = -\frac{\tanh(\Sigma_t x/\upsilon_o)}{(1-c)\upsilon_o}J(x) + \frac{\Sigma_t/\upsilon_o}{2\sinh(\Sigma_t/2\upsilon_o)\cosh(\Sigma_t x/\upsilon_o)}\bar{\phi} - T(x) \quad (21)$$

where T(x) is a function involving only integrals over the continuous transient modes. Evaluating this equation at x=1/2 or -1/2, the following relations are obtained among the surface average currents, surface average fluxes and the node average flux,

$$J^L = 2\frac{D}{C^j}\left(\phi_s^L - C^f\bar{\Phi} + T^L\right) \quad (22a)$$

$$J^R = -2\frac{D}{C^j}\left(\phi_s^R - C^f\bar{\Phi} + T^R\right) \quad (22b)$$

where

$$C^j = \frac{\tanh(\Sigma_t/2\upsilon_o)}{\Sigma_t/2\upsilon_o} \quad (23a)$$

$$C^f = \frac{\Sigma_t/\upsilon_o}{\sinh(\Sigma_t/\upsilon_o)} \quad (23b)$$

$$D = \frac{(1-c)\upsilon_o^2}{\Sigma_t} \quad (24)$$

Eqs.(22) looks the same as eqs.(9) except for the additional term of T, but it is of exactly the same form as the corresponding equation in the multi-dimensional diffusion case where T is from the transverse leakage term (see Ref.5). Therefore the same process as in Ref.5 can be used to show that the generic results for CMFD, i.e. eqs.(11) and (13), are also valid for the transport

case. So in using CMFD the only difference in the transport case from the diffusion case is that the basic coefficient, C^j, is defined by eqs.(23a) instead of eqs.(10a), and D is defined by eq.(24).

A even closer analogy to the diffusion case is revealed if one makes use of eq.(24) to rewrite the argument of the hyperbolic functions in eqs.(23) as follows,

$$k = (\Sigma_t / \upsilon_o) = \frac{\sqrt{(1-c)}}{\sqrt{D/\Sigma_t}} = \sqrt{(1-c)}/L \tag{25}$$

where L is the diffusion length. This equation is the same as eq.(10c), as the pitch distance has been set to one, the parameter c becomes k-infinity in the case of one energy group, and the hyperbolic functions are used instead of the trigonometric functions. In the limit of c approaching one, the transport results reduce to the diffusion results. In this limit, eq.(16) reduces to

$$\lim_{c \to 1} (1-c)\upsilon_o^2 \to \frac{1}{3} \tag{26}$$

Applying this limit to eq.(24), the diffusion coefficient reduces to the classical value of the inverse of $3\Sigma_{t.}$

4.2 CASE WITH FIXED SOURCE

Up to this point, in both Ref.5 and this paper, we have assumed only fission source, without fixed extraneous source. In the calculation of various coefficients that appear in the definition of EDC, the k-infinity (in Ref.5) or c (in this paper) contains the fission source contribution. This can be done for the case of one group and without fixed source. For the case of multi-group or with fixed source, there appears in the equation for each group an extraneous source term. For the case of two group diffusion equation without fixed source, this "apparent" extraneous source term is "removed" by diagonalization in the energy group space (see Ref.5). In transport calculation, this diagonalization can not be performed. Thus for more than one group transport calculation, we will always have an "apparent" extraneous source term. The appearance of the extraneous source term is as if it were a fixed source term, except that with a real fixed source the core has to be subcritical. The following discussion and treatment of the fixed source applies to both real fixed source and apparent fixed source in either diffusion or transport case.

As mentioned below eq.(14), the particular solution associated with the constant fixed source term can be trivially obtained. With the fixed source particular solution included, we go back to revisit the one dimensional diffusion case and find that eqs.(9a) and (9b) become,

$$J = -2\frac{D_n}{C_n^j}\left(\phi_s - C_n^f \bar{\Phi}_n - C_n^s S_n\right) \tag{27a}$$

$$J = 2\frac{D_{n+1}}{C^j_{n+1}}\left(\phi_s - C^f_{n+1}\bar{\Phi}_{n+1} - C^s_{n+1}S_{n+1}\right) \qquad (27b)$$

where

$$C^s = \left(\frac{C^f - 1}{k^2}\right)\left(\frac{a^2}{D}\right) \qquad (28)$$

and *a* is the pitch distance which has been set to 1 throughout this paper. Like eqs.(9) being exact for the one dimensional diffusion case without fixed source, eqs.(27) is exact for the one dimensional diffusion case with fixed source. Now we see something interesting. Recalling eq.(10c) and the discussion following it, eqs.(9) reduces to the classical finite difference limit of eqs.(1) in the limit of either k-infinity approaching 1 or the pitch approaching 0. However in eq.(28), as *k* approaches zero the first factor in C^s approaches -1/6 and C^s does not approach zero unless *a* approaches zero. Therefore, if a is not very small (coarse mesh), unlike eqs(9), eqs.(27) does not approach the classical finite difference eq.(1) regardless of the value of k-infinity. In order to keep the CMFD formulation completely rigorous for the one dimensional diffusion case, the CMFD current relation will be modified as follows to accommodate the fixed source term from eqs.(27).

Eliminating the interface flux or current in eqs.(27), one finds that the exact CMFD relations for the one dimensional diffusion case with fixed source are modified from eqs.(11a) and (11b) by adding to them the fixed source contributions, $(J)_{fs}$ and $(\phi)_{fs}$,

$$J = -\left[D^L_{n+1}\ \bar{\Phi}_{n+1}\ - D^R_n\bar{\Phi}_n\right] + (J)_{fs} \qquad (29a)$$

$$\phi_s = C^j_n\ \frac{1}{2D_n}D^L_{n+1}\ \bar{\Phi}_{n+1}\ +\ C^j_{n+1}\ \frac{1}{2D_{n+1}}D^R_n\bar{\Phi}_n + (\phi_s)_{fs} \qquad (29b)$$

where $(J)_{fs}$ and $(\phi)_{fs}$ are given by,

$$(J)_{fs} = -\left[F^L_{n+1}\ S_{n+1}\ - F^R_n S_n\right] \qquad (30a)$$

$$(\phi_s)_{fs} = C^j_n\ \frac{1}{2D_n}F^L_{n+1}\ S_{n+1}\ +\ C^j_{n+1}\ \frac{1}{2D_{n+1}}F^R_n S_n \qquad (30b)$$

where the directional dependent F is given as

$$F^L_n = 2\left[C^j_n\frac{1}{D_n} + C^j_{n-1}\frac{1}{D_{n-1}}\right]^{-1}C^s_n \qquad (31a)$$

$$F_n^R = 2\left[C_n^j \frac{1}{D_n} + C_{n+1}^j \frac{1}{D_{n+1}}\right]^{-1} C_n^s \qquad (31b)$$

It is not difficult to show that the EDCs defined in eqs.(13) for general application beyond the one dimensional case should be modified to include the fixed source contribution as follows,

$$D_n^L = 2\left[C_n^j \frac{1}{D_n} + C_{n-1}^j \frac{1}{D_{n-1}}\right]^{-1}\left[\frac{\Phi_n^L}{\bar{\Phi}_n} - C_n^j \frac{J_n^L}{2D_n\bar{\Phi}_n}\right] - F_n^L\left(\frac{S_n}{\bar{\Phi}_n}\right) \qquad (32a)$$

$$D_n^R = 2\left[C_n^j \frac{1}{D_n} + C_{n+1}^j \frac{1}{D_{n+1}}\right]^{-1}\left[\frac{\Phi_n^R}{\bar{\Phi}_n} + C_n^j \frac{J_n^R}{2D_n\bar{\Phi}_n}\right] - F_n^R\left(\frac{S_n}{\bar{\Phi}_n}\right) \qquad (32b)$$

Eqs.(29) to (32) provide the complete general CMFD formulation for the case with fixed source. They apply to both diffusion and transport calculations, with the only difference that the "diffusion" coefficient in the transport case is defined by eq. (24). The definition of *k* in the diffusion or transport case is self-consistent with the use of trigonometric or hyperbolic functions in the respective cases.

It should be noted that the CMFD current relation (29a) contains the additional constant term, $(J)_{fs,}$ contributed from the fixed source. When this CMFD current relation is used in the neutron balance relation to establish the CMFD matrix equation, the $(J)_{fs}$ term should be moved to the source term side to be combined with the fixed source in the neutron balance relation.

5. CONCLUSIONS

Via a unified treatment of various CMFD formulations, analysis and comparisons of the different formulations are made. The CMFD and ACMFD formulation recently proposed by the author is shown to be theoretically more rigorous and should numerically be more stable. Both the CMFD and the ACMFD methods have been implemented in the Westinghouse core design diffusion code ANC as additional options. Their performances are found to be very effective and robust. The simplicity and good accuracy of ACMFD is very attractive for loading pattern search applications.The CMFD formulation can be generalized for transport calculations as well, resulting in a form identical to that for the diffusion case except for a different definition of the diffusion coefficient. For fixed source calculations, the CMFD formulation needs a modification to include in the surface current additional terms that are contributed by the fixed source.

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