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Derivation of the Space and Energy Dependent Formula for the Third Order Neutron Correlation Technique

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We have studied a measurement technique of subcriticality by using the neutron correlation methods. Among various techniques, we have paid attention to the third order neutron correlation technique, which utilizes the third order fluctuations of neutron counts. By using this technique, we can obtain the absolute value of subcriticality without prior knowledge of the prompt neutron decay constant at a reference state. To apply the third order neutron correlation technique to actual experiments, we must consider the effects of the spatial and neutron energy distributions in this technique. For this purpose, we derived the generalized theoretical formulas of the third order neutron correlation technique that took account of the spatial and neutron energy effects.

KEYWORDS: subcriticality, absolute value, neutron correlation methods, third order neutron correlation, space effect, energy dependence

1. Introduction

From the viewpoint of the criticality safety, measurement of the subcriticality is one of the most important subjects. Among various techniques, we have studied the variance-to-mean (V-to-M) technique, i.e. the Feynman-α technique.[1] The V-to-M technique does not require special experimental equipments, such as the pulsed neutron source or the ^{252}Cf installed ionization chamber, except for a stationary extraneous neutron source and an ordinary neutron detector.

In the V-to-M technique, the second order neutron correlation factor Y is measured from the V-to-M ratio of the number of neutron counts. By analyzing the measured Y value, we can estimate the prompt neutron decay constant α which contains the information of subcriticality. However, to obtain the absolute value of the subcriticality, one must convert the obtained α value to the absolute subcriticality, by comparing with the α value estimated beforehand at a reference state, such as a critical state.

To overcome this problem, we have paid attention to the third order neutron correlation technique, i.e. the Furuhashi's technique, which utilizes the third order fluctuations of neutron counts.[2, 3] In this technique, in addition to the conventional second order neutron correlation factor Y, the third order neutron correlation factor X is measured from the neutron counts data. According to the theoretical formulas derived by Furuhashi and Izumi, we can obtain the absolute subcriticality from the saturation values of Y and X, without prior knowledge of α value at a reference state.

Through the experiments in the Kyoto University Critical Assembly, we performed measurement of the subcriticality by using the third order neutron correlation technique.[4] It was shown that the results of this technique were different from the reference values which were measured by the area ratio technique of pulsed neutron method. It seemed that the difference was caused by the effects of spatial and neutron energy distributions, because the results of subcriticality were evaluated from the theoretical formulas derived by using the one-point reactor approximation.

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Therefore, we derived generalized theoretical formulas of the third order neutron correlation technique that took account of the spatial and neutron energy effects. In this paper, we will describe their derivations.

2. Outline of third order neutron correlation technique

In the V-to-M technique, the second order neutron correlation factor Y is defined as follows:

$$Y(T) \equiv \frac{v(T)}{m(T)} - 1, \tag{1}$$

where the notations of $m(T)$ and $v(T)$ represent the mean and the variance of neutron counts C during counting gate width T, respectively. The definitions of $m(T)$ and $v(T)$ are as follows:

$$m(T) \equiv \langle C(T) \rangle, \tag{2}$$

$$v(T) \equiv \left\langle (C(T) - m(T))^2 \right\rangle, \tag{3}$$

where bracket $\langle x \rangle$ means the expected value of x. Assuming that a steady-state subcritical system is maintained by an extraneous stationary neutron source, we use the one-point reactor approximation without delayed neutrons. Then, we can derive the simple theoretical formula of Y as follows:

$$Y(T) = Y_\infty \left(1 - \frac{1 - \mathrm{e}^{-\alpha T}}{\alpha T} \right), \tag{4}$$

$$\alpha = \mathrm{v} \langle \nu \rangle \Sigma_\mathrm{f} (-\rho), \tag{5}$$

$$Y_\infty = \frac{\Sigma_\mathrm{d}}{\Sigma_\mathrm{f}} \frac{\langle \nu(\nu - 1) \rangle}{\langle \nu \rangle^2} \frac{1 + \delta_2 (-\rho)}{(-\rho)^2}, \tag{6}$$

$$\delta_2 \equiv \frac{\langle \nu \rangle \langle q(q-1) \rangle}{\langle q \rangle \langle \nu(\nu - 1) \rangle}, \tag{7}$$

where

Y_∞	=	saturation value of Y when T goes to infinity,
α	=	neutron decay constant,
$(-\rho)$	=	subcriticality,
v	=	neutron velocity,
Σ_f	=	macroscopic fission cross section,
Σ_d	=	macroscopic detection cross section,
ν	=	number of neutrons emitted by one fission event,
q	=	number of neutrons emitted by one extraneous source event.

By fitting Eq. (4) to measured $Y(T)$, we can evaluate both α and Y_∞. We can find from Eqs. (5) and (6) that the absolute value of subcriticality cannot be determined directly from measured Y_∞ and α even if all the factorial moments of ν and q are available.

To overcome this problem, Furuhashi and Izumi proposed the third order neutron correlation factor X.[2] The X value is defined as follows:

$$X(T) \equiv \frac{s(T)}{m(T)} - 3\frac{v(T)}{m(T)} + 2, \tag{8}$$

where the notation of $s(T)$ represents the third order central moment of neutron counts during counting gate width T. The definition of $s(T)$ is as follows:

$$s(T) \equiv \left\langle (C(T) - m(T))^3 \right\rangle. \tag{9}$$

Assuming the same approximation for the *Y* value, we can derive the simple theoretical formula of *X* as follows:

$$X(T) = X_{2\infty}\left(1+\mathrm{e}^{-\alpha T} - 2\frac{1-\mathrm{e}^{-\alpha T}}{\alpha T}\right) + X_{3\infty}\left(1-\frac{3-4\mathrm{e}^{-\alpha T}+\mathrm{e}^{-2\alpha T}}{2\alpha T}\right), \tag{10}$$

$$X_{2\infty} = 3\left(\frac{\Sigma_{\mathrm{d}}}{\Sigma_{\mathrm{f}}}\frac{\langle\nu(\nu-1)\rangle}{\langle\nu\rangle^2}\right)^2 \frac{1+\delta_2(-\rho)}{(-\rho)^4}, \tag{11}$$

$$X_{3\infty} = \left(\frac{\Sigma_{\mathrm{d}}}{\Sigma_{\mathrm{f}}}\right)^2 \frac{\langle\nu(\nu-1)(\nu-2)\rangle}{\langle\nu\rangle^3}\frac{1+\delta_3(-\rho)}{(-\rho)^3}, \tag{12}$$

$$\delta_3 \equiv \frac{\langle\nu\rangle\langle q(q-1)(q-2)\rangle}{\langle q\rangle\langle\nu(\nu-1)(\nu-2)\rangle}. \tag{13}$$

In Eq. (10), the saturation value of *X* when *T* goes to infinity, namely X_∞, is

$$X_\infty = X_{2\infty} + X_{3\infty}. \tag{14}$$

By using Eqs. (6) and (14), we can obtain the relationship between the saturation values of *Y* and *X*:

$$\lim_{T\to\infty}\left(\frac{X(T)}{Y(T)^2}\right) = \frac{X_\infty}{Y_\infty^2} = \frac{3}{1+\delta_2(-\rho)} + F\frac{\{1+\delta_3(-\rho)\}(-\rho)}{\{1+\delta_2(-\rho)\}^2}, \tag{15}$$

$$F \equiv \frac{\langle\nu\rangle\langle\nu(\nu-1)(\nu-2)\rangle}{\langle\nu(\nu-1)\rangle^2}. \tag{16}$$

By using Eq. (15), we can evaluate the absolute value of subcriticality from the saturation values of *Y* and *X*, if statistics of ν and *q*, namely δ_2, δ_3, and *F*, are known.

3. Generalized theory of third order neutron correlation technique

To apply the third order neutron correlation technique to actual experiments, we must consider the spatial and neutron energy effects in this technique. For this purpose, we derived the generalized theoretical formulas of this technique as follows. Firstly, by using the eigenfunction expansion, we derived the Green function, which expresses the angular neutron density due to one neutron born at the certain position, energy, direction and time. Secondly, by using this Green function, we derived the single-, pair-, and trio-detection probabilities based on a heuristic method. These single-, pair-, and trio-detection probabilities mean the probabilities for detecting one, a pair, and a trio of neutrons respectively. Thirdly, by using these detection probabilities, we can obtain the factorial moments of neutron counts. Finally, from these factorial moments, we obtained the generalized theoretical formulas of third order neutron correlation technique.

3.1 Green Function

Let us denote as Green function $G(\boldsymbol{r},E,\boldsymbol{\Omega},t\,|\,\boldsymbol{r}_0,E_0,\boldsymbol{\Omega}_0,t_0)$ to express the angular neutron density at $(\boldsymbol{r},E,\boldsymbol{\Omega},t)$ due to one neutron born at $(\boldsymbol{r}_0,E_0,\boldsymbol{\Omega}_0,t_0)$. The Green function $G(\boldsymbol{r},E,\boldsymbol{\Omega},t\,|\,\boldsymbol{r}_0,E_0,\boldsymbol{\Omega}_0,t_0)$ is defined as follows:

$$\frac{\partial G}{\partial t} = -\boldsymbol{B}\,G(\boldsymbol{r},E,\boldsymbol{\Omega},t\,|\,\boldsymbol{r}_0,E_0,\boldsymbol{\Omega}_0,t_0) + \delta(\boldsymbol{r}-\boldsymbol{r}_0)\delta(E-E_0)\delta(\boldsymbol{\Omega}-\boldsymbol{\Omega}_0)\delta(t-t_0), \tag{17}$$

where ***B*** is the Boltzmann operator,

$$\boldsymbol{B} \equiv \mathrm{v}(E)\boldsymbol{\Omega}\nabla + \mathrm{v}(E)\Sigma_{\mathrm{t}}(\boldsymbol{r},E) - \int_0^\infty dE' \int_{4\pi} d\boldsymbol{\Omega}'\ \mathrm{v}(E')\Sigma_{\mathrm{s}}(\boldsymbol{r},E' \to E, \boldsymbol{\Omega}' \to \boldsymbol{\Omega})$$
$$- \frac{\chi_{\mathrm{f}}(E)}{4\pi} \int_0^\infty dE' \int_{4\pi} d\boldsymbol{\Omega}' \mathrm{v}(E')\Sigma_{\mathrm{f}}(\boldsymbol{r},E') \sum_{\nu=0}^{\infty} \nu\, p_{\mathrm{f}}(\nu, E'), \quad (18)$$

where

$\mathrm{v}(E)$ = neutron velocity,

$\Sigma_{\mathrm{t}}(\boldsymbol{r},E)$ = macroscopic total cross section,

$\Sigma_{\mathrm{s}}(\boldsymbol{r},E' \to E, \boldsymbol{\Omega}' \to \boldsymbol{\Omega})$ = macroscopic scattering cross section from the energy E' and the direction $\boldsymbol{\Omega}'$ to the energy E and the direction $\boldsymbol{\Omega}$,

$\chi_{\mathrm{f}}(E)$ = fission spectrum,

$\Sigma_{\mathrm{f}}(\boldsymbol{r},E)$ = macroscopic fission cross section,

$p_{\mathrm{f}}(\nu,E)$ = probability that ν neutrons are emitted by a fission event induced by a neutron of the energy E.

To solve Eq. (17), we use the eigenfunction expansion.[5, 6] It is achieved by using the α-eigenvalue equations as follows:

$$\boldsymbol{B}\psi_n(\boldsymbol{r},E,\boldsymbol{\Omega}) = \alpha_n \psi_n(\boldsymbol{r},E,\boldsymbol{\Omega}), \quad (19)$$
$$\boldsymbol{B}^\dagger \psi_n^\dagger(\boldsymbol{r},E,\boldsymbol{\Omega}) = \alpha_n \psi_n^\dagger(\boldsymbol{r},E,\boldsymbol{\Omega}), \quad (20)$$

where

$\boldsymbol{B}^\dagger$ = adjoint operator of $\boldsymbol{B}$,

$$\boldsymbol{B}^\dagger \equiv -\mathrm{v}(E)\boldsymbol{\Omega}\nabla + \mathrm{v}(E)\Sigma_{\mathrm{t}}(\boldsymbol{r},E) - \int_0^\infty dE' \int_{4\pi} d\boldsymbol{\Omega}'\ \mathrm{v}(E)\Sigma_{\mathrm{s}}(\boldsymbol{r},E \to E', \boldsymbol{\Omega} \to \boldsymbol{\Omega}')$$
$$- \mathrm{v}(E)\Sigma_{\mathrm{f}}(\boldsymbol{r},E) \sum_{\nu=0}^{\infty} \nu\, p_{\mathrm{f}}(\nu,E) \int_0^\infty dE' \int_{4\pi} d\boldsymbol{\Omega}' \frac{\chi_{\mathrm{f}}(E')}{4\pi}, \quad (21)$$

α_n = neutron decay constant of n-th mode,

$\psi_n(\boldsymbol{r},E,\boldsymbol{\Omega})$ = eigenfunction corresponding to the eigenvalue α_n,

$\psi_n^\dagger(\boldsymbol{r},E,\boldsymbol{\Omega})$ = adjoint eigenfunction corresponding to the eigenvalue α_n.

Here, the eigenfunctions $\psi_n(\boldsymbol{r},E,\boldsymbol{\Omega})$ and $\psi_n^\dagger(\boldsymbol{r},E,\boldsymbol{\Omega})$ satisfy the orthonormal condition,

$$\int_V d\boldsymbol{r} \int_0^\infty dE \int_{4\pi} d\boldsymbol{\Omega}\, \psi_m^\dagger(\boldsymbol{r},E,\boldsymbol{\Omega})\psi_n(\boldsymbol{r},E,\boldsymbol{\Omega}) = \delta_{mn}. \quad (22)$$

By using Eqs. (19), (20) and (22), we can derive the Green function $G(\boldsymbol{r},E,\boldsymbol{\Omega},t\,|\,\boldsymbol{r}_0,E_0,\boldsymbol{\Omega}_0,t_0)$ as the expanded form by these eigenfunctions as follows:

$$G(\boldsymbol{r},E,\boldsymbol{\Omega},t\,|\,\boldsymbol{r}_0,E_0,\boldsymbol{\Omega}_0,t_0) = \sum_n \psi_n(\boldsymbol{r},E,\boldsymbol{\Omega})\,\psi_n^\dagger(\boldsymbol{r}_0,E_0,\boldsymbol{\Omega}_0)\,\mathrm{e}^{-\alpha_n(t-t_0)}. \quad (23)$$

3.2 Single-, pair-, and trio-detection probabilities

3.2.1 Single-detection probability

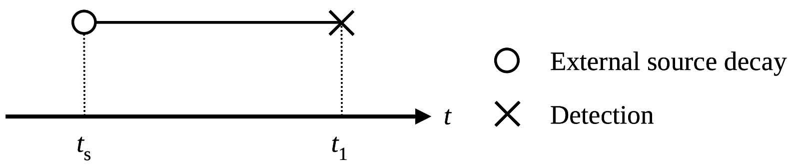

Fig. 1 Single-detection process

As shown in Fig. 1, the single-detection probability $P_1(t_1)\,dt_1$ means the probability that one neutron is detected during the time interval dt_1 around the time t_1. Suppose that one neutron is detcted at t_1 after an external source event at time t_s, the probability $P_1(t_1)\,dt_1$ is constructed by the following elemental process:

1. Probability that an external source decays at the position $\boldsymbol{r}_s$ and the time t_s: $S(\boldsymbol{r}_s)\,d\boldsymbol{r}_s dt_s$, where $S(\boldsymbol{r})$ is the spatial distribution of external neutron source strength.
2. Probability that q neutrons are emitted by this external source event: $p_s(q)$.
3. Probability that a neutron of the energy E_s is emitted toward direction $\boldsymbol{\Omega}_s$ by this external source event: $\left(\chi_s(E_s)/4\pi\right)dE_s d\boldsymbol{\Omega}_s$, where $\chi_s(E)$ is the energy spectrum of extraneous source.
4. Probability that one of all descendants due to these q neutrons born at $(\boldsymbol{r}_s, E_s, \boldsymbol{\Omega}_s, t_s)$ exists at $(\boldsymbol{r}_1, E_1, \boldsymbol{\Omega}_1, t_1)$: $q\,G(\boldsymbol{r}_1, E_1, \boldsymbol{\Omega}_1, t_1 \mid \boldsymbol{r}_s, E_s, \boldsymbol{\Omega}_s, t_s)\,d\boldsymbol{r}_1 dE_1 d\boldsymbol{\Omega}_1$.
5. Probability that the descendant is detected at $(\boldsymbol{r}_1, E_1, \boldsymbol{\Omega}_1, t_1)$: $\mathrm{v}(E_1)\Sigma_d(\boldsymbol{r}_1, E_1)\,dt_1$, where $\Sigma_d(\boldsymbol{r}, E)$ is the macroscopic detection cross section.

By integrating the product of these elemental processes over the specific range, we can derive $P_1(t_1)\,dt_1$ as follows:

$$\begin{aligned} P_1(t_1)\,dt_1 &= \int_{-\infty}^{t_1} dt_s \int_V d\boldsymbol{r}_s\, S(\boldsymbol{r}_s) \sum_{q=0}^{\infty} q\, p_s(q) \int_0^{\infty} dE_s \int_{4\pi} d\boldsymbol{\Omega}_s \frac{\chi_s(E_s)}{4\pi} \\ &\quad \times \int_V d\boldsymbol{r}_1 \int_0^{\infty} dE_1 \int_{4\pi} d\boldsymbol{\Omega}_1\, \mathrm{v}(E_1)\Sigma_d(\boldsymbol{r}_1, E_1) \sum_n \psi_n(\boldsymbol{r}_1, E_1, \boldsymbol{\Omega}_1)\, \psi_n^{\dagger}(\boldsymbol{r}_s, E_s, \boldsymbol{\Omega}_s)\, \mathrm{e}^{-\alpha_n (t_1 - t_s)}\, dt_1 \\ &= \sum_n \frac{S_n D_n}{\alpha_n} dt_1 , \end{aligned} \tag{24}$$

where

$$S_n \equiv \int_V d\boldsymbol{r}\, S(\boldsymbol{r}) \sum_{q=0}^{\infty} q\, p_s(q)\, \Psi_{s,n}^{\dagger}(\boldsymbol{r}) , \tag{25}$$

$$\Psi_{s,n}^{\dagger}(\boldsymbol{r}) \equiv \int_0^{\infty} dE \int_{4\pi} d\boldsymbol{\Omega} \frac{\chi_s(E)}{4\pi} \psi_n^{\dagger}(\boldsymbol{r}, E, \boldsymbol{\Omega}) , \tag{26}$$

$$D_n \equiv \int_V d\boldsymbol{r} \int_0^{\infty} dE \int_{4\pi} d\boldsymbol{\Omega}\, \mathrm{v}(E)\Sigma_d(\boldsymbol{r}, E)\, \psi_n(\boldsymbol{r}, E, \boldsymbol{\Omega}) . \tag{27}$$

3.2.2 Pair-detection probability

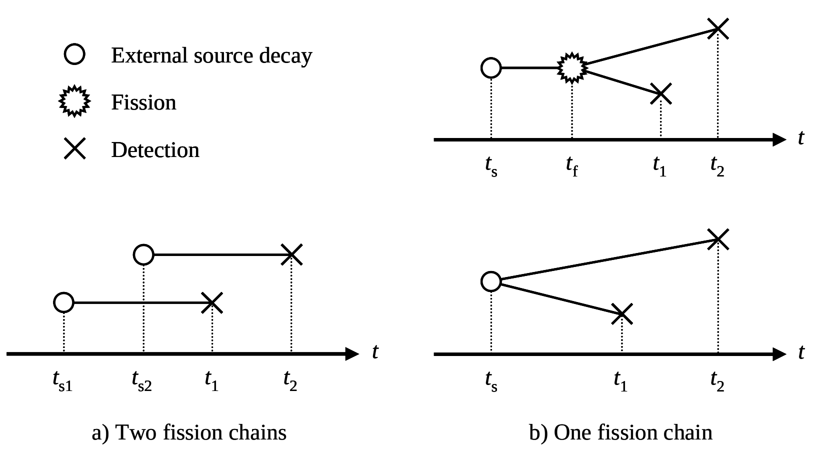

Fig. 2 Pair-detection processes

As shown in Fig. 2, the pair-detection probability $P_2(t_1,t_2)\,dt_1dt_2$ means the probability that a pair of neutrons is detected during the small time intervals *dt* around the time t_1 and t_2 ($t_1 < t_2$). This probability is divided into two terms; one is an uncorrelated term and the other is a correlated term.

The uncorrelated term of the pair-detection probability is caused by the two independent fission chains. Therefore the uncorrelated term of this probability is expressed by $P_1(t_1)\,P_1(t_2)\,dt_1dt_2$.

On the other hand, the correlated term of this probability is caused by the same fission chain. Furthermore, the correlated term is composed of two cases. In one case of this correlated term, the fission chain forks by a fission source event. This is constructed by the following elemental processes:

1. Probability that an external source decays at the position $\boldsymbol{r}_\mathrm{s}$ and the time t_s: $S(\boldsymbol{r}_\mathrm{s})\,d\boldsymbol{r}_\mathrm{s}dt_\mathrm{s}$.
2. Probability that q neutrons are emitted by this external source event: $p_\mathrm{s}(q)$.
3. Probability that a neutron of the energy E_s is emitted toward direction $\boldsymbol{\Omega}_\mathrm{s}$ by this external source event: $\left(\chi_\mathrm{s}(E_\mathrm{s})/4\pi\right)dE_\mathrm{s}d\boldsymbol{\Omega}_\mathrm{s}$.
4. Probability that one of all descendants due to these q neutrons born at $(\boldsymbol{r}_\mathrm{s},E_\mathrm{s},\boldsymbol{\Omega}_\mathrm{s},t_\mathrm{s})$ exists at $(\boldsymbol{r}_\mathrm{f},E_\mathrm{f},\boldsymbol{\Omega}_\mathrm{f},t_\mathrm{f})$: $q\,G(\boldsymbol{r}_\mathrm{f},E_\mathrm{f},\boldsymbol{\Omega}_\mathrm{f},t_\mathrm{f}\,|\,\boldsymbol{r}_\mathrm{s},E_\mathrm{s},\boldsymbol{\Omega}_\mathrm{s},t_\mathrm{s})\,d\boldsymbol{r}_\mathrm{f}dE_\mathrm{f}d\boldsymbol{\Omega}_\mathrm{f}$.
5. Probability that the descendant causes a fission reaction: $\mathrm{v}(E_\mathrm{f})\,\Sigma_\mathrm{f}(\boldsymbol{r}_\mathrm{f},E_\mathrm{f})\,dt_\mathrm{f}$.
6. Probability that ν neutrons are emitted by this fission event: $p_\mathrm{f}(\nu,E_\mathrm{f})$.
7. Probability that a neutron of the energy E'_f is emitted toward the direction $\boldsymbol{\Omega}'_\mathrm{f}$ by this fission event: $\left(\chi_\mathrm{f}(E'_\mathrm{f})/4\pi\right)dE'_\mathrm{f}d\boldsymbol{\Omega}'_\mathrm{f}$.
8. Probability that one of all descendants due to these ν neutrons born at $(\boldsymbol{r}_\mathrm{f},E'_\mathrm{f},\boldsymbol{\Omega}'_\mathrm{f},t_\mathrm{f})$ exists at $(\boldsymbol{r}_1,E_1,\boldsymbol{\Omega}_1,t_1)$: $\nu\,G(\boldsymbol{r}_1,E_1,\boldsymbol{\Omega}_1,t_1\,|\,\boldsymbol{r}_\mathrm{f},E'_\mathrm{f},\boldsymbol{\Omega}'_\mathrm{f},t_\mathrm{f})\,d\boldsymbol{r}_1dE_1d\boldsymbol{\Omega}_1$.
9. Probability that the descendant is detected at $(\boldsymbol{r}_1,E_1,\boldsymbol{\Omega}_1,t_1)$: $\mathrm{v}(E_1)\,\Sigma_\mathrm{d}(\boldsymbol{r}_1,E_1)\,dt_1$.
10. Probability that another descendant due to the other $(\nu-1)$ neutrons born at $(\boldsymbol{r}_\mathrm{f},E''_\mathrm{f},\boldsymbol{\Omega}''_\mathrm{f},t_\mathrm{f})$ exists at $(\boldsymbol{r}_2,E_2,\boldsymbol{\Omega}_2,t_2)$: $(\nu-1)G(\boldsymbol{r}_2,E_2,\boldsymbol{\Omega}_2,t_2\,|\,\boldsymbol{r}_\mathrm{f},E''_\mathrm{f},\boldsymbol{\Omega}''_\mathrm{f},t_\mathrm{f})\,d\boldsymbol{r}_2dE_2d\boldsymbol{\Omega}_2$.
11. Probability that the descendant is detected at $(\boldsymbol{r}_2,E_2,\boldsymbol{\Omega}_2,t_2)$: $\mathrm{v}(E_2)\,\Sigma_\mathrm{d}(\boldsymbol{r}_2,E_2)\,dt_2$.

In the other case of the correlated term, the fission chain forks by an external source event. This is also constructed by the following elemental processes:

1. Probability that an external source decays at the position $\boldsymbol{r}_\mathrm{s}$ and the time t_s: $S(\boldsymbol{r}_\mathrm{s})\,d\boldsymbol{r}_\mathrm{s}dt_\mathrm{s}$
2. Probability that q neutrons are emitted by this external source event: $p_\mathrm{s}(q)$.
3. Probability that a neutron of the energy E_s is emitted toward the direction $\boldsymbol{\Omega}_\mathrm{s}$ by this external source event: $\left(\chi_\mathrm{s}(E_\mathrm{s})/4\pi\right)dE_\mathrm{s}d\boldsymbol{\Omega}_\mathrm{s}$.
4. Probability that one of all descendants due to these q neutrons born at $(\boldsymbol{r}_\mathrm{s},E_\mathrm{s},\boldsymbol{\Omega}_\mathrm{s},t_\mathrm{s})$ exists at $(\boldsymbol{r}_1,E_1,\boldsymbol{\Omega}_1,t_1)$: $q\,G(\boldsymbol{r}_1,E_1,\boldsymbol{\Omega}_1,t_1\,|\,\boldsymbol{r}_\mathrm{s},E_\mathrm{s},\boldsymbol{\Omega}_\mathrm{s},t_\mathrm{s})\,d\boldsymbol{r}_1dE_1d\boldsymbol{\Omega}_1$.
5. Probability that the descendant is detected at $(\boldsymbol{r}_1,E_1,\boldsymbol{\Omega}_1,t_1)$: $\mathrm{v}(E_1)\,\Sigma_\mathrm{d}(\boldsymbol{r}_1,E_1)\,dt_1$.
6. Probability that another descendant due to the other $(q-1)$ neutrons born at $(\boldsymbol{r}_\mathrm{s},E'_\mathrm{s},\boldsymbol{\Omega}'_\mathrm{s},t_\mathrm{s})$ exists at $(\boldsymbol{r}_2,E_2,\boldsymbol{\Omega}_2,t_2)$: $(q-1)G(\boldsymbol{r}_2,E_2,\boldsymbol{\Omega}_2,t_2\,|\,\boldsymbol{r}_\mathrm{s},E'_\mathrm{s},\boldsymbol{\Omega}'_\mathrm{s},t_\mathrm{s})\,d\boldsymbol{r}_2dE_2d\boldsymbol{\Omega}_2$.
7. Probability that the descendant is detected at $(\boldsymbol{r}_2,E_2,\boldsymbol{\Omega}_2,t_2)$: $\mathrm{v}(E_2)\,\Sigma_\mathrm{d}(\boldsymbol{r}_2,E_2)\,dt_2$.

Thus, the total probability for detecting a pair of neutrons, $P_2(t_1,t_2)\,dt_1dt_2$, is derived as follows:

$$P_2(t_1,t_2)\,dt_1dt_2 = \left[P_1(t_1)\,P_1(t_2) + \sum_m\sum_n\left\{\left(\sum_l \frac{S_l F_{l\to mn}}{\alpha_l}\right) + S_{mn}\right\}\frac{D_m D_n}{\alpha_m+\alpha_n}\,\mathrm{e}^{-\alpha_n(t_2-t_1)}\right]dt_1dt_2\,, \tag{28}$$

where

$$F_{l\to mn} \equiv \int_V d\boldsymbol{r} \int_0^\infty dE \int_{4\pi} d\boldsymbol{\Omega}\, \mathrm{v}(E)\Sigma_\mathrm{f}(\boldsymbol{r},E)\psi_l(\boldsymbol{r},E,\boldsymbol{\Omega}) \sum_{\nu=0}^{\infty} \nu(\nu-1) p_\mathrm{f}(\nu,E)\Psi^\dagger_{\mathrm{f},m}(\boldsymbol{r})\Psi^\dagger_{\mathrm{f},n}(\boldsymbol{r}) \,, \tag{29}$$

$$\Psi^\dagger_{\mathrm{f},n}(\boldsymbol{r}) \equiv \int_0^\infty dE \int_{4\pi} d\boldsymbol{\Omega} \frac{\chi_\mathrm{f}(E)}{4\pi} \psi^\dagger_n(\boldsymbol{r},E,\boldsymbol{\Omega}) \,, \tag{30}$$

$$S_{mn} \equiv \int_V d\boldsymbol{r}\, S(\boldsymbol{r}) \sum_{q=0}^{\infty} q(q-1) p_\mathrm{s}(q)\Psi^\dagger_{\mathrm{s},m}(\boldsymbol{r})\Psi^\dagger_{\mathrm{s},n}(\boldsymbol{r}) \,. \tag{31}$$

3.2.3 Trio-detection probability

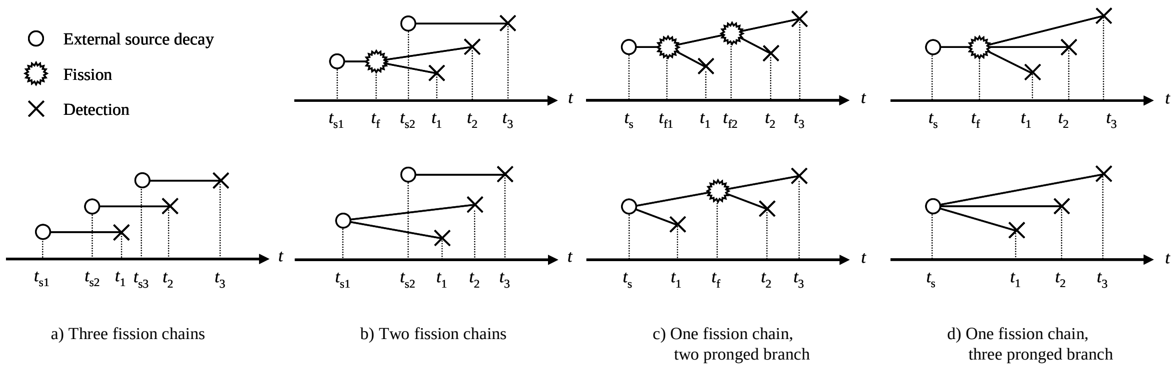

Fig. 3 Trio-detection processes

As shown in Fig. 3, trio-detection probability $P_3(t_1,t_2,t_3)\,dt_1dt_2dt_3$ means the probability that a trio of neutrons is detected during the small time intervals dt around the time t_1, t_2 and t_3 ($t_1 < t_2 < t_3$).

By applying the same method of the derivations for $P_1(t_1)\,dt_1$ and $P_2(t_1,t_2)\,dt_1dt_2$ to seven processes shown in Fig. 3, we can derive $P_3(t_1,t_2,t_3)\,dt_1dt_2dt_3$ as follows:

$$\begin{aligned}
&P_3(t_1,t_2,t_3)\,dt_1dt_2dt_3 \\
&= \left[\begin{aligned}
&P_1(t_1)P_1(t_2)P_1(t_3) \\
&+P_1(t_1)\big(P_2(t_2,t_3)-P_1(t_2)P_1(t_3)\big) \\
&+P_1(t_2)\big(P_2(t_1,t_3)-P_1(t_1)P_1(t_3)\big) \\
&+P_1(t_3)\big(P_2(t_1,t_2)-P_1(t_1)P_1(t_2)\big) \\
&+\sum_k\sum_l\sum_m\sum_n \left\{\left(\sum_j \frac{S_jF_{j\to kl}}{\alpha_j}\right)+S_{kl}\right\} \frac{F_{k\to mn}D_lD_mD_n}{-\alpha_k+\alpha_m+\alpha_n}\left(\frac{\mathrm{e}^{-\alpha_k(t_2-t_1)-\alpha_n(t_3-t_2)}}{\alpha_k+\alpha_l}-\frac{\mathrm{e}^{-\alpha_m(t_2-t_1)-\alpha_n(t_3-t_1)}}{\alpha_l+\alpha_m+\alpha_n}\right) \\
&+\sum_k\sum_l\sum_m\sum_n \left\{\left(\sum_j \frac{S_jF_{j\to km}}{\alpha_j}\right)+S_{km}\right\} \frac{F_{k\to ln}D_lD_mD_n}{(\alpha_k+\alpha_m)(\alpha_l+\alpha_m+\alpha_n)}\mathrm{e}^{-\alpha_m(t_2-t_1)-\alpha_n(t_3-t_1)} \\
&+\sum_k\sum_l\sum_m\sum_n \left\{\left(\sum_j \frac{S_jF_{j\to kn}}{\alpha_j}\right)+S_{kn}\right\} \frac{F_{k\to lm}D_lD_mD_n}{(\alpha_k+\alpha_n)(\alpha_l+\alpha_m+\alpha_n)}\mathrm{e}^{-\alpha_m(t_2-t_1)-\alpha_n(t_3-t_1)} \\
&+\sum_l\sum_m\sum_n \left\{\left(\sum_k \frac{S_kF_{k\to lmn}}{\alpha_k}\right)+S_{lmn}\right\} \frac{D_lD_mD_n}{\alpha_l+\alpha_m+\alpha_n}\mathrm{e}^{-\alpha_m(t_2-t_1)-\alpha_n(t_3-t_1)}
\end{aligned}\right] \\
&\times dt_1dt_2dt_3 \,,
\end{aligned} \tag{32}$$

where

$$F_{k\to lmn} \equiv \int_V d\boldsymbol{r}\int_0^\infty dE\int_{4\pi} d\boldsymbol{\Omega}\, \mathrm{v}(E)\Sigma_\mathrm{f}(\boldsymbol{r},E)\psi_k(\boldsymbol{r},E,\boldsymbol{\Omega})\sum_{\nu=0}^{\infty}\nu(\nu-1)(\nu-2)\,p_\mathrm{f}(\nu,E)\,\Psi^\dagger_{\mathrm{f},l}(\boldsymbol{r})\Psi^\dagger_{\mathrm{f},m}(\boldsymbol{r})\Psi^\dagger_{\mathrm{f},n}(\boldsymbol{r})\ , \quad (33)$$

$$S_{lmn} \equiv \int_V d\boldsymbol{r}\, S(\boldsymbol{r})\sum_{q=0}^{\infty} q(q-1)(q-2)\,p_\mathrm{s}(q)\,\Psi^\dagger_{\mathrm{s},l}(\boldsymbol{r})\Psi^\dagger_{\mathrm{s},m}(\boldsymbol{r})\Psi^\dagger_{\mathrm{s},n}(\boldsymbol{r})\ . \quad (34)$$

3.3 Factorial moments of neutron counts

Firstly, by using the single-detection probability $P_1(t_1)\,dt_1$, the mean of neutron counts C during the counting gate width T, i.e. $\langle C(T)\rangle$, is derived as follows:

$$\langle C(T)\rangle = \int_0^T dt_1\, P_1(t_1) = \sum_n \frac{S_n D_n}{\alpha_n} T = C_\mathrm{R}\, T\ , \quad (35)$$

$$C_\mathrm{R} \equiv \sum_n \frac{S_n D_n}{\alpha_n}\ , \quad (36)$$

where C_R means the neutron count rate.

Secondary, by using the pair-detection probability $P_2(t_1,t_2)\,dt_1 dt_2$, the expected number of pairs detected during T, $\langle C(T)(C(T)-1)/2\rangle$ is derived as follows:

$$\left\langle \frac{C(T)(C(T)-1)}{2}\right\rangle = \int_0^T dt_2 \int_0^{t_2} dt_1\, P_2(t_1,t_2)\ . \quad (37)$$

Finally, by using the trio-detection probability $P_3(t_1,t_2,t_3)\,dt_1 dt_2 dt_3$, the expected number of neutron trios detected during T, $\langle C(T)(C(T)-1)(C(T)-2)/6\rangle$ is derived as follows:

$$\left\langle \frac{C(T)(C(T)-1)(C(T)-2)}{6}\right\rangle = \int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, P_3(t_1,t_2,t_3)\ . \quad (38)$$

3.4 Theoretical formulas of *Y* and *X*

By using Eqs. (35) and (37), we can derive the theoretical formula of Y as follows:

$$\begin{aligned} Y(T) &= \frac{\langle C(T)(C(T)-1)\rangle - \langle C(T)\rangle^2}{\langle C(T)\rangle} \\ &= \frac{2}{C_\mathrm{R}}\sum_m\sum_n\left\{\left(\sum_l \frac{S_l F_{l\to mn}}{\alpha_l}\right) + S_{mn}\right\}\frac{D_m D_n}{\alpha_n(\alpha_m+\alpha_n)}\left(1-\frac{1-\mathrm{e}^{-\alpha_n T}}{\alpha_n T}\right). \end{aligned} \quad (39)$$

From Eq. (39), we can obtain the saturation value of Y when T goes to infinity, i.e. Y_∞, as follows:

$$\begin{aligned} Y_\infty &= \lim_{T\to\infty} Y(T) \\ &= \frac{2}{C_\mathrm{R}}\sum_m\sum_n\left\{\left(\sum_l \frac{S_l F_{l\to mn}}{\alpha_l}\right) + S_{mn}\right\}\frac{D_m D_n}{\alpha_n(\alpha_m+\alpha_n)} \\ &= \frac{1}{C_\mathrm{R}}\sum_m\sum_n\left\{\left(\sum_l \frac{S_l F_{l\to mn}}{\alpha_l}\right) + S_{mn}\right\}\frac{D_m D_n}{\alpha_m\alpha_n}\left(1+\frac{\alpha_m-\alpha_n}{\alpha_m+\alpha_n}\right) \\ &= \frac{1}{C_\mathrm{R}}\sum_m\sum_n\left\{\left(\sum_l \frac{S_l F_{l\to mn}}{\alpha_l}\right) + S_{mn}\right\}\frac{D_m D_n}{\alpha_m\alpha_n}. \end{aligned} \quad (40)$$

The last expression of Eq. (40) can be derived by using a symmetry of $(\alpha_m - \alpha_n)$.

By using Eq. (35), (37) and (38), we can also derive the theoretical formula of X as follows:

$$
\begin{aligned}
X(T) &= \frac{\langle C(T)(C(T)-1)(C(T)-2)\rangle - 3\langle C(T)(C(T)-1)\rangle\langle C(T)\rangle + 2\langle C(T)\rangle^3}{\langle C(T)\rangle} \\
&= \frac{6}{C_{\mathrm{R}}}\sum_k\sum_l\sum_m\sum_n\left\{\left(\sum_j \frac{S_j F_{j\to kl}}{\alpha_j}\right)+S_{kl}\right\}\frac{F_{k\to mn}D_l D_m D_n}{\alpha_k\alpha_n(\alpha_k+\alpha_l)(-\alpha_k+\alpha_m+\alpha_n)} \\
&\qquad \times\left\{\begin{aligned}&1-\frac{1}{\alpha_k T}-\frac{1-\mathrm{e}^{-\alpha_n T}}{\alpha_n T} \\ &+\left(1+\frac{1}{\alpha_n T}\right)\mathrm{e}^{-\alpha_n T}\delta_{kn}+\frac{\alpha_k \mathrm{e}^{-\alpha_n T}-\alpha_n \mathrm{e}^{-\alpha_k T}}{\alpha_k(\alpha_k-\alpha_n)T}(1-\delta_{kn})\end{aligned}\right\} \\
&+\frac{6}{C_{\mathrm{R}}}\sum_k\sum_l\sum_m\sum_n\left[\begin{aligned}&\left\{\left(\sum_j \frac{S_j F_{j\to kl}}{\alpha_j}\right)+S_{kl}\right\}\frac{F_{k\to mn}}{\alpha_k-\alpha_m-\alpha_n} \\ &+\left\{\left(\sum_j \frac{S_j F_{j\to km}}{\alpha_j}\right)+S_{km}\right\}\frac{F_{k\to ln}}{\alpha_k+\alpha_m}+\left\{\left(\sum_j \frac{S_j F_{j\to kn}}{\alpha_j}\right)+S_{kn}\right\}\frac{F_{k\to lm}}{\alpha_k+\alpha_n}\end{aligned}\right] \\
&\qquad \times\frac{D_l D_m D_n}{\alpha_n(\alpha_m+\alpha_n)(\alpha_l+\alpha_m+\alpha_n)}\left\{1-\frac{1-\mathrm{e}^{-\alpha_m T}}{\alpha_m T}-\frac{1-\mathrm{e}^{-\alpha_n T}}{\alpha_n T}+\frac{\alpha_n}{\alpha_m}\frac{1-\mathrm{e}^{-(\alpha_m+\alpha_n)T}}{(\alpha_m+\alpha_n)T}\right\} \\
&+\frac{6}{C_{\mathrm{R}}}\sum_l\sum_m\sum_n\left\{\left(\sum_k \frac{S_k F_{k\to lmn}}{\alpha_k}\right)+S_{lmn}\right\}\frac{D_l D_m D_n}{\alpha_n(\alpha_m+\alpha_n)(\alpha_l+\alpha_m+\alpha_n)} \\
&\qquad \times\left\{1-\frac{1-\mathrm{e}^{-\alpha_m T}}{\alpha_m T}-\frac{1-\mathrm{e}^{-\alpha_n T}}{\alpha_n T}+\frac{\alpha_n}{\alpha_m}\frac{1-\mathrm{e}^{-(\alpha_m+\alpha_n)T}}{(\alpha_m+\alpha_n)T}\right\}. \qquad (41)
\end{aligned}
$$

From Eq. (41), we can obtain the saturation value of X when T goes to infinity, i.e. X_∞, as follows:

$$
\begin{aligned}
X_\infty &= \lim_{T\to\infty} X(T) \\
&= \frac{1}{C_{\mathrm{R}}}\left[\begin{aligned}&3\sum_k\sum_l\sum_m\sum_n\left\{\left(\sum_j \frac{S_j F_{j\to kl}}{\alpha_j}\right)+S_{kl}\right\}\frac{F_{k\to mn}D_l D_m D_n}{\alpha_k\alpha_l\alpha_m\alpha_n} \\ &+\sum_l\sum_m\sum_n\left\{\left(\sum_k \frac{S_k F_{k\to lmn}}{\alpha_k}\right)+S_{lmn}\right\}\frac{D_l D_m D_n}{\alpha_l\alpha_m\alpha_n}\end{aligned}\right]. \qquad (42)
\end{aligned}
$$

To derived Eq. (42), we used symmetries of $(\alpha_l-\alpha_m)$, $(\alpha_l-\alpha_n)$ and $(\alpha_m-\alpha_n)$ in the same way of Eq. (40).

3.5 Fundamental mode approximation

The generalized theoretical formulas of Y_∞ and X_∞, Eq. (40) and (42), are too complicated, because these theoretical formulas are expressed by the sum of infinite series of ψ_n and $\psi_n^\dagger$. To obtain the practical formulas, we take only the fundamental modes ψ_0 and $\psi_0^\dagger$ into account. Then, saturation values Y_∞ and X_∞ are approximated as follows:

$$
Y_\infty \cong \left(\frac{F_{0\to 00}}{\alpha_0}+\frac{S_{00}}{S_0}\right)\frac{D_0}{\alpha_0}, \qquad (43)
$$

$$
X_\infty \cong 3\left(\frac{F_{0\to 00}}{\alpha_0}+\frac{S_{00}}{S_0}\right)\frac{F_{0\to 00}}{\alpha_0}\left(\frac{D_0}{\alpha_0}\right)^2+\left(\frac{F_{0\to 000}}{\alpha_0}+\frac{S_{000}}{S_0}\right)\left(\frac{D_0}{\alpha_0}\right)^2. \qquad (44)
$$

Here, the fundamental eigenvalue α_0 is expressed by using the subcriticality $(-\rho)$ as follows:

$$\alpha_0 = \frac{(-\rho)}{\Lambda}. \tag{45}$$

where Λ is the neutron generation time. By using Eqs. (43), (44) and (45), we can obtain the relationship between Y_∞ and X_∞ as follows:

$$\frac{X_\infty}{Y_\infty^2} \cong \frac{3}{1+\delta_{2,0}(-\rho)} + F_0 \frac{\{1+\delta_{3,0}(-\rho)\}(-\rho)}{\{1+\delta_{2,0}(-\rho)\}^2}, \tag{46}$$

$$\delta_{2,0} \equiv \frac{S_{00}}{S_0}\frac{1/\Lambda}{F_{0\to 00}}, \tag{47}$$

$$\delta_{3,0} \equiv \frac{S_{000}}{S_0}\frac{1/\Lambda}{F_{0\to 000}}, \tag{48}$$

$$F_0 \equiv \frac{F_{0\to 000}/\Lambda}{(F_{0\to 00})^2}. \tag{49}$$

As compared with Eq. (15) in the case of one-point reactor approximation, the coefficients $\delta_{2,0}$, $\delta_{3,0}$, and F_0, include the spatial and neutron energy effects through the fundamental modes ψ_0 and $\psi_0^\dagger$. Therefore, we should note that the expression of fundamental mode approximation is different from one of the one-point reactor approximation. This characteristic is well known in the Rossi-α technique based on the second order neutron correlation.[7] Through the derivations of the generalized third order neutron correlation technique, we can point out that the third order neutron correlation technique has the same characteristic. If we make use of the generalized coefficients $\delta_{2,0}$, $\delta_{3,0}$, and F_0 estimated by the numerical calculations, we can evaluate the absolute value of subcriticality $(-\rho)$ by using the generalized formula Eq. (40).

4. Conclusion

To apply the third order neutron correlation technique to actual experiments, we must consider the spatial and neutron energy effects in this technique. For this purpose, we derived the generalized theoretical formulas of the third order neutron correlation technique that took account of the spatial and neutron energy effects. By applying these generalized formulas to actual experiments, we expect that the absolute value of subcriticality is rigorously evaluated. We will verify this generalized third order neutron correlation technique by analysis of the experiments or the numerical calculations in the future.

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