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Theoretical and Practical Study of the Variance and Efficiency of a Monte Carlo calculation due to Russian roulette

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Although Russian roulette is applied very often in Monte Carlo calculations, not much literature exists on its quantitative influence on the variance and efficiency of a Monte Carlo calculation. Elaborating on the work of Lux and Koblinger using moment equations, new relevant equations are derived to calculate the variance of a Monte Carlo simulation using Russian roulette. To demonstrate its practical application the theory is applied to a simplified transport model resulting in explicit analytical expressions for the variance of a Monte Carlo calculation and for the expected number of collisions per history. From these expressions numerical results are shown and compared with actual Monte Carlo calculations, showing an excellent agreement. By considering the number of collisions in a Monte Carlo calculation as a measure of the CPU time, also the efficiency of the Russian roulette can be studied. It opens the way for further investigations, including optimization of Russian roulette parameters.

KEYWORDS: Monte Carlo, Russian roulette, variance, moment equations

1. Introduction

In Monte Carlo calculations for radiation transport the implicit capture technique is used very often, i.e. at every collision the particle weight is adapted to account for absorption. Then the Russian roulette technique is needed to cut off histories of particles with low statistical weight. Although this generally increases the variance of the Monte Carlo process, it reduces the computer time spent on histories, which does not seem very important anymore for the result aimed at. Taken together, application of Russian roulette may increase the efficiency of the Monte Carlo process. No matter how often Russian roulette is applied in Monte Carlo calculations, little is known about the quantitative influence on the variance and efficiency of the calculation. This paper derives relevant equations and demonstrates their application and verification for a simplified transport problem. It opens the way for further investigations, including the optimization of Russian roulette parameters.

2. Theoretical Calculation of the Variance due to Russian Roulette

For analog Monte Carlo calculations and classes of non-analog Monte Carlo calculations using biased transport kernels, an equation can be derived [1,2] for the second moment of the score, from which the variance can be obtained. However, this theory does not apply for a game using Russian roulette, as this depends on the statistical weight of the particle. Lux and Koblinger [3] presented a general theoretical framework for deriving an equation for the second moment of a Monte Carlo score, introducing the score probability distribution. This

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theory includes the possibility of splitting and Russian roulette, but it was not worked out for the usual case that the probability for Russian roulette (or splitting) depends on the particle weight. Moreover, their derivation refers to a particle leaving the source or a collision. It turns out that when considering particles entering a collision, the problem is much better to deal with and the equations become simpler.

2.1 Forward and Adjoint Transport Equations

The transport of neutrons and photons is mostly described in terms of the particle flux ϕ for which an integro-differential equation can be written down. However, introducing the emission density $\chi(\boldsymbol{r},E,\boldsymbol{\Omega})$ such that $\chi(\boldsymbol{r},E,\boldsymbol{\Omega})dEd\Omega$ is the expected number of particles coming out of the source or a collision with energy E in dE and direction $\boldsymbol{\Omega}$ in the solid angle $d\Omega$, the emission density is given by

$$\chi(P) = S(P) + \iint C(\boldsymbol{r}, E' \to E, \boldsymbol{\Omega}' \to \boldsymbol{\Omega})\psi(\boldsymbol{r}, E', \boldsymbol{\Omega}')dE'd\Omega', \tag{1}$$

with P=($\boldsymbol{r}$,E,$\boldsymbol{\Omega}$) a point in the phase space, $S(P)$ the particle source density and $\psi(P)$ the collision density, which is connected to the emission density by

$$\psi(P) = \int T(\boldsymbol{r}' \to \boldsymbol{r}, E, \boldsymbol{\Omega})\chi(\boldsymbol{r}', E, \boldsymbol{\Omega})dV'. \tag{2}$$

In these equations C is the collision kernel, which is normalized to the non-absorption probability $\kappa(P')$

$$\kappa(P') = \int C(\boldsymbol{r}', E' \to E, \boldsymbol{\Omega}' \to \Omega)dEd\boldsymbol{\Omega}. \tag{3}$$

T is the transition kernel, which is normalized to 1 for an infinite medium. Both equations can be combined to an integral equation for the emission density χ featuring the particle source S. This route is followed by Lux and Koblinger. [3] As noted by these authors, both equations can also be combined to an integral equation for the collision density ψ. Which approach is taken should not matter for the final result of the variance of the calculation, but taking the equation for the collision density as the starting point is more natural when considering a Russian roulette event, which takes place at a collision and this approach leads to simpler formulas. In this case we need the source of first collisions S_1

$$S_1(P) = \int T(\boldsymbol{r}' \to \boldsymbol{r}, E, \boldsymbol{\Omega})S(\boldsymbol{r}', E, \boldsymbol{\Omega})dV', \tag{4}$$

leading to the following equation for $\psi(P)$

$$\psi(P) = S_1(P) + \int L(P' \to P)\psi(P')dP'. \tag{5}$$

The kernel L is the product of the collision and transport kernels

$$L(P' \to P) = C(\boldsymbol{r}', E' \to E, \boldsymbol{\Omega}' \to \boldsymbol{\Omega})T(\boldsymbol{r}' \to \boldsymbol{r}, E, \boldsymbol{\Omega}) \tag{6}$$

Note that when sampling the kernel L the collision kernel C is sampled first and the transit kernel T afterwards.

The aim of the Monte Carlo calculation is a (hypothetical) detector response, which can be described as an integral over the collision density

$$R = \int g(P)\psi(P)dP. \tag{7}$$

In a Monte Carlo simulation this means that a score $g(P)$ is applied when having a sample of the collision density, that is, when applying a collision estimator.

In the following we will also need the solution of the equation adjoint to Eq.(5) given by

$$\psi^+(P) = g(P) + \int L^+(P' \to P)\psi^+(P')dP'. \tag{8}$$

with the adjoint kernel $L^+(P' \to P) = L(P \to P')$. Now the detector response can also be obtained from $R = \int S_1(P)\,\psi^+(P)\mathrm{d}P$.

2.2 Russian Roulette

As the collision kernel C is not normalized to unity, the normalization factor or non-absorption probability $\kappa(P)$ must be taken into account. To reduce the variance, this is often done by introducing a statistical weight for each particle and multiplying this weight at every collision by κ. This, however, leads to infinitely long histories with smaller and smaller weights. To cut off the histories with particles having a relatively low weight, which are not expected to contribute significantly any more to the detector response, a Russian roulette is played. With a certain probability the history is ended and otherwise the weight is increased in such a way that on the average the particle weight from before the Russian roulette is conserved.

Although a Russian roulette can be realized in different ways, we consider here the following form. If the particle weight W is below a threshold value $W_{th}(P)$ then with probability $z_1(P)$ the particle survives the Russian roulette game and is assigned a weight $W_{sv}(P)$. With probability $z_0(P)=1-z_1(P)$ the particle is killed and the history ended. Naturally, $W_{sv}(P) > W_{th}(P)$. For a statistically fair game $z_1(P)=W/W_{sv}(P)$. Note that the Russian roulette parameters may all be space, energy and direction dependent. Moreover, the Russian roulette game is played before the collision kernel C is sampled for a new energy and direction and after the score $Wg(P)$ is taken for the collision estimator and the particle weight is multiplied by the non-absorption probability κ.

To include the case that no Russian roulette is played we define

$$z_1(P,W) = \begin{cases} \dfrac{W}{W_{sv}(P)} & W \le W_{th}(P) \\ 1 & W > W_{th}(P) \end{cases} \tag{9}$$

$$z_0(P,W) = 1 - z_1(P,W) = \begin{cases} 1 - \dfrac{W}{W_{sv}(P)} & W \le W_{th}(P) \\ 0 & W > W_{th}(P) \end{cases} \tag{10}$$

with the particle weight after the Russian roulette equal to

$$W^*(P,W) = \begin{cases} W_{sv}(P) & W \le W_{th}(P) \\ W & W > W_{th}(P) \end{cases} \quad (11)$$

2.3 The Probability Score Equation

In line with Lux and Koblinger [3] we define the score probability $\pi(P,W,s)\mathrm{d}s$ as the probability of a particle at P with statistical weight W now going into a collision (instead of leaving a collision or the source) will yield, together with its progeny, a total score in $\mathrm{d}s$ around s. Next, we have to take into account the Russian roulette procedure explicitly. Although it is not very difficult to generalize the following procedure to include for instance splitting - as already shown by Lux and Koblinger - we concentrate here on Russian roulette in order not to complicate formulas more than necessary to point out the basic idea.

For a particle with weight W going into a collision at P the score is $Wg(P)$. Next the particle weight is multiplied by the non-absorption probability $\kappa(P)$ to become κW. Hence, the Russian roulette is actually played for a particle weight κW. If the Russian roulette is played and the particle is killed in the Russian roulette the score remains $Wg(P)$. If it survives, the normalized collision kernel

$$C_n(\boldsymbol{r}, E \to E', \boldsymbol{\Omega} \to \boldsymbol{\Omega}') = C(\boldsymbol{r}, E \to E', \boldsymbol{\Omega} \to \boldsymbol{\Omega}') / \kappa(P) \quad (12)$$

is sampled for a new energy E' and direction $\boldsymbol{\Omega}'$ and subsequently the transition kernel $T(\boldsymbol{r} \to \boldsymbol{r}', E', \boldsymbol{\Omega}')$ for a new collision site $\boldsymbol{r}'$. Then a score $W^*g(P')$ is added to the history score so far. Hence, the score probability is composed of two term, one for the case of the particle is killed by the Russian roulette and another when it survives the Russian roulette.

$$\pi(P,W,s) = z_0(P,\kappa W)\delta(s - Wg(P)) + z_1(P,\kappa W)\int L_n(P \to P')\pi(P', W^*(P,W), s - Wg(P))dP'. \quad (13)$$

with $L_n(P \to P') = L(P \to P')/\kappa(P)$ the normalized transport kernel, equal to the product of C_n and T.

2.4 Derivation of the Moment Equations

The first moment of the score probability gives the expected score of a particle with weight W going into a collision at P. Multiplication of Eq.(13) by s and integration over all s leads in the second term on the right hand side to an integral

$$\int_{-\infty}^{\infty} s\pi(P', W^*, s - Wg(P))ds = \int_{-\infty}^{\infty} (s' + Wg(P))\pi(P', W^*, s')ds = M_1(P', W^*) + Wg(P) \quad (14)$$

Hence,

$$\begin{aligned} M_1(P,W) &= \int_{-\infty}^{\infty} s\pi(P,W,s)ds \\ &= z_0(P,\kappa W)Wg(P) + z_1(P,\kappa W)\left[\int L_n(P \to P')M_1(P', W^*)dP' + Wg(P)\right] \\ &= Wg(P) + \int L_n(P \to P')M_1(P', W^*)dP' \end{aligned} \quad (15)$$

Under fairly general conditions it holds that [3] $M_1(P,W) = WM_1(P,1) = WM_1(P)$. Taking into account Eqs.(9) and (11) with the actual weight κW of the particle subject to Russian roulette, Eq.(15) becomes

$$M_1(P) = g(P) + \kappa(P)\int L_n(P \to P')M_1(P')dP' = g(P) + \int L(P \to P')M_1(P')dP'. \quad (16)$$

From Eq.(8) it becomes clear that $M_1(P)$ is equal to the adjoint function $\psi^{+}(P)$, which is to be expected as both functions represent the expected contribution to the detector response for a particle with unit weight going into a collision at P.

Multiplication of Eq.(13) by s^2 and integration gives the second moment. Further manipulation, using the adjoint equation of the particle collision density, leads to

$$M_2(P,W) = W^2 g(P)\{2M_1(P) - g(P)\} + z_1(P,\kappa W)\int L_n(P \to P')M_2(P',W^*)dP' \quad (17)$$

Solution of the equation for the second moment is more complicated than solving the adjoint equation as in general $M_2(P,W) \neq W^2 M_2(P)$ and the moments M_2 in the left hand and right hand side appear at different values of the particle weight. Nonetheless, this equation can be solved for simplified cases, as will be shown below.

The variance of the contribution of a particle with weight W going into a collision at P is obtained from

$$V(P,W) = M_2(P,W) - M_1^2(P,W). \quad (18)$$

The variance of the total detector response is given by

$$V_R = \int S_1(P)V(P,W=1)dP. \quad (19)$$

Note that the moments M_n differ from the moments defined by Lux and Koblinger as they now refer to particles going into a collision instead of coming out of a collision or the source.

2.5 Number of Collisions as a Measure of the Efficiency

The efficiency of a Monte Carlo calculation is often defined as the inverse of the product of the variance and the computer time of the calculation. As the computer time is very difficult to fit in general parameters, we adopt here the average number of collisions for a particle entering a collision at P with weight W, which can be obtained from the equation

$$N_c(P,W) = 1 + z_1(P,\kappa W)\int L_n(P \to P')N_c(P',W^*)dP'. \quad (20)$$

When multiplied by the variance $V(P,W)$, optimization of the calculation with respect to the Russian roulette parameters can be sought. The expected number of collisions for a source particle is

$$N_C = \int S_1(P)N_C(P,W=1)dP. \quad (21)$$

For a further discussion on how to determine the efficiency of a Monte Carlo calculation, see Glyn and Witt. [4]

3. Practical Application

3.1 Two-direction Transport Model

It will be clear that such rather complicated equations as for the second moment of the score cannot be solved analytically in practical cases. We therefore adopt a simplified transport model: the monoenergetic transport of particles that can move only in the $+X$ and $-X$ direction. For this model the particle flux can be exactly described by a diffusion equation with the diffusion coefficient being equal to the inverse total cross section. [5]

$$\frac{d}{dx}\frac{1}{\Sigma_t(x)}\frac{d\phi(x)}{dx} - \Sigma_a(x)\phi(x) = S(x). \tag{22}$$

This allows the analytical calculation of the particle flux and other related quantities as a function of x. We consider an infinite homogeneous system with total cross section Σ_t, scattering probability $\kappa=\Sigma_s/\Sigma_t$ and a normalized source density

$$S(x) = \frac{1}{2\alpha}e^{-\alpha|x|} \tag{23}$$

The solution of this equation in terms of the collision density is

$$\psi(x) = \frac{1}{2}\frac{\alpha\Sigma_t^2}{\Sigma_a\Sigma_t - \alpha^2}\left\{e^{-\alpha|x|} - \frac{\alpha}{\sqrt{\Sigma_a\Sigma_t}}e^{-\sqrt{\Sigma_a\Sigma_t}|x|}\right\}. \tag{24}$$

The collision kernel C degenerates to $\frac{1}{2}\Sigma_s/\Sigma_t$, with the factor ½ arising from the equal scattering probability in either direction. The transport kernel simplifies to

$$T(x' \to x) = \Sigma_t e^{-\Sigma_t|x'-x|}. \tag{25}$$

The source of first collision can be calculated from $S(x)$:

$$S_1(x) = \int T(x' \to x)S(x')dx' = \frac{1}{2}\frac{\alpha\Sigma_t}{\Sigma_a\Sigma_t - \alpha^2}\left\{\Sigma_t e^{-\alpha|x|} - \alpha e^{-\Sigma_t|x|}\right\}. \tag{26}$$

As the aim of the calculation we choose the averaged flux for the range $|x|\leq a$. Hence, the scoring function $g(x)=1/(2a\Sigma_t)$ for $|x|\leq a$ and zero elsewhere. This leads to a detector response

$$R = \int g(x)\psi(x)dx = \frac{1}{2a\Sigma_a}\left\{1 + \frac{\alpha^2}{\Sigma_a\Sigma_t - \alpha^2}\Sigma_t e^{-\sqrt{\Sigma_a\Sigma_t}a} - \frac{\Sigma_a\Sigma_t}{\Sigma_a\Sigma_t - \alpha^2}e^{-\alpha a}\right\}. \tag{27}$$

As the diffusion equation is self-adjoint, the adjoint function can, in principle, be calculated from solving Eq.(22) with the particle source S replaced by the detector response function. However, then the adjoint function for particles leaving a collision or the source is obtained. Here we need the adjoint function for particles going into a collision, which should be obtained from Eq.(8). As it is more easy to solve a differential equation than an integral equation, the integral equations of the type of Eq.(5) or (8) can be transformed to a second-order differential equation of the form of Eq.(22), but with a different source term. In our case the differential equation equivalent to Eq.(8) is

$$\frac{d^2\psi^+(x)}{dx^2} - \Sigma_a\Sigma_t\psi^+(x) = \frac{d^2 g(x)}{dx^2} - \Sigma_t^2 g(x). \tag{28}$$

which gives the solution for the adjoint function

$$\psi^+(x) = \frac{\Sigma_s}{2a\Sigma_a\Sigma_t}\begin{cases}\Sigma_t/\Sigma_s - e^{-\sqrt{\Sigma_a\Sigma_t}a}\cosh\sqrt{\Sigma_a\Sigma_t}x & |x|\le a \\ \sinh\sqrt{\Sigma_a\Sigma_t}a\ e^{-\sqrt{\Sigma_a\Sigma_t}|x|} & |x|> a\end{cases} \tag{29}$$

3.2 Solution of the Second-moment Equation

To solve the second-moment equation we first solve it for $W=W_{sv}$, as in this case the particle weight always returns to this value after surviving the Russian roulette. If we further assume $\kappa W_{sv}\le W_{th}$ then $z_1(x,\kappa W_{sv})=\kappa W/W_{sv}$ and from Eq.(17)

$$M_2(x,W_{sv}) = W_{sv}^2 g(x)\{2M_1(x)-g(x)\} + \kappa\int L_n(x\to x')M_2(x',W_{sv})dx' \tag{30}$$

This equation is of the same type as Eq.(16) but with a different source term and can also be transformed into a differential equation like Eq.(28). From this differential equation we can see that the solution for $M_2(x,W_{sv})$ for $|x|>a$ will behave as $\exp(-\sqrt{\Sigma_a\Sigma_t}\,|x|)$. For $|x|\le a$ the solution of the inhomogeneous differential equation is $\cosh\sqrt{\Sigma_a\Sigma_t}\,x$. As the source term due to the presence of $M_1(x)$ also generates this solution, the complete solution is

$$M_2(x,W_{sv}) = \begin{cases} A_{sv} + B_{sv}x\sinh\sqrt{\Sigma_a\Sigma_t}x + C_{sv}\cosh\sqrt{\Sigma_a\Sigma_t}x & |x|\le a \\ E_{sv}e^{-\sqrt{\Sigma_a\Sigma_t}|x|} & |x|> a\end{cases} \tag{31}$$

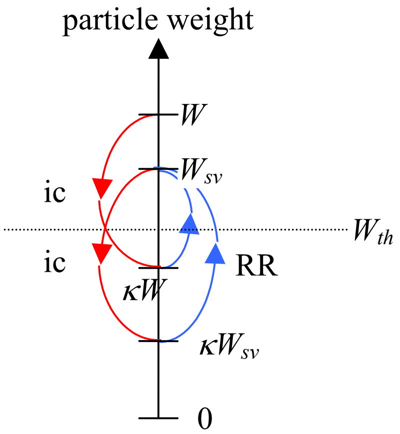

Fig. 1 Particle weight changes due to implicit capture (ic) and Russian roulette (RR)

The constants can be determined by substitution in the differential or the integral equation.

Next, the second-moment equation can be solved for particle weights $W\le W_{th}/\kappa$, as in this case the particle weight falls below the threshold after accounting for implicit capture (see Fig. 1) and the Russian roulette will always be played, leading to a weight W_{sv} after surviving. As we solved already the second-moment equation for $W=W_{sv}$ we get an explicit expression

$$\begin{aligned} M_2(x,W) &= W^2 g(x)\{2M_1(x)-g(x)\} + \kappa\frac{W}{W_{sv}}\int L_n(x\to x')M_2(x',W_{sv})dx' \\ &(W^2 - WW_{sv})g(x)\{2M_1(x)-g(x)\} + \frac{W}{W_{sv}}M_2(x,W_{sv}) \end{aligned} \qquad W\le W_{th}/\kappa \tag{32}$$

where Eq.(30) has been used to eliminate the integral term. The solution has the same form as in Eq.(31) but with different constants.

When considering particle weights $W_{th}/\kappa \leq W \leq W_{th}/\kappa^2$ the particle will not be subjected to Russian roulette the first time after accounting for implicit capture. Hence, $z_1(x,\kappa W)=1$ and $W^*=\kappa W$. Then the function $M_2(x,W^*)$ in the integral of the right hand side of Eq.(17) refers to the second moment for a particle weight range we just solved. This process can be repeated for higher weights, where we need the integrals

$$I_n(x) = \int L(x \to x')I_{n-1}(x')dx' \quad n = 1,2,... \tag{33}$$

with

$$I_0(x) = g(x)\{2M_1(x) - g(x)\} \tag{34}$$

Note that Eq.(33) can be transformed into a differential equation:

$$\frac{d^2 I_n(x)}{dx^2} - \Sigma_t^2 I_n(x) = -\Sigma_t^2 I_{n-1}(x). \tag{35}$$

Then it can be seen that the functions $I_n(x)$ that the solution is

$$I_n(x) = \begin{cases} A_n + B_n \cosh\sqrt{\Sigma_a \Sigma_t}\, x + \sum_{k=0}^{n} x^k (C_{nk} \cosh \Sigma_t x + D_{nk} \sinh \Sigma_t x) & |x| \leq a \\ \sum_{k=0}^{n} E_{nk} |x|^k \exp(-\Sigma_t |x|) & |x| > a \end{cases} \tag{36}$$

with recurrent relations for the coefficients A_n, B_n, C_{nk}, D_{nk}, and E_{nk}. As $I_n(x)$ are even functions the coefficients C_{nk} are zero for odd k and D_{nk} are zero for even k. Moreover, noting that $M_1(x)=\psi^+(x)$ we have for n=0 from Eq.(34) and Eq.(29) $C_{00}=D_{00}=0$.

With the functions $I_n(x)$ the second moment can be written as

$$M_2(x,W)/W^2 = \frac{W_{sv}}{W} M_2(x,W_{sv}) + \sum_{i=0}^{n} \kappa^i (\kappa^i - \frac{W_{sv}}{W}) I_i(x) \quad W_{th}/\kappa^n \leq W < W_{th}/\kappa^{n+1} \tag{37}$$

In this derivation we assumed $\kappa W_{sv} \leq W_{th}$. If κ is closer to unity a particle will suffer more collisions before its weight will be below the threshold value and the roulette played. This situation can also be included in the theory.

For the situation that the Russian roulette is never played, z_1=1, independent of the particle weight, and $W^*=\kappa W$. In this case holds $M_2(P,W)=W^2 M_2(P)$. [3] Then it follows from Eq.(17)

$$M_2(P) = \frac{M_2(P,W)}{W^2} = g(P)\{2M_1(P) - g(P)\} + \kappa(P)\int L_n(P \to P')M_2(P')dP'. \tag{38}$$

This integral equation for our one-dimensional case can also be transformed to a differential equation like Eq.(28), but because of the factor κ in front of the integral term in Eq.(38), the coefficient $\Sigma_a \Sigma_t$ in the differential equation is replaced by $\sqrt{\Sigma_t^2 - \Sigma_s^2}$. Therefore, the solution of Eq.(38) is

$$M_{2no}(x) = \begin{cases} A_{no} + B_{no}\cosh\sqrt{\Sigma_a \Sigma_t}\,x + C_{no}\cosh\sqrt{\Sigma_t^2 - \Sigma_s^2}\,x & |x| \le a \\ E_{no} e^{-\sqrt{\Sigma_t^2 - \Sigma_s^2}|x|} & |x| > a \end{cases} \quad (39)$$

In this case a particle history will never be ended, but this case can be approximated in practice using a very low Russian roulette threshold value.

3.3 Number of Collision per History

If the survival probability for the Russian roulette is independent of position, the number of collisions is also independent of the position the particle is having a collision. However, it remains dependent on the particle weight. Therefore, we look first to the situation with $W=W_{sv}$. From Eq.(20) we have

$$N_c(W_{sv}) = \frac{1}{1-\kappa} = \frac{\Sigma_t}{\Sigma_a}. \quad (40)$$

As for the second moment we can now determine the number of collisions for successive particle weight ranges, leading to

$$N_c(W) = n + \frac{W}{W_{sv}} \frac{\kappa^n}{1-\kappa} \qquad \text{with } n = 1 + [\frac{\ln W_{th}/W}{\ln \kappa}]. \quad (41)$$

4. Numerical Application

4.1 Solution of the Second-moment Equation

We applied the theory derived above in a numerical simulation, using a total cross section Σ_t=1 cm^{-1} and scattering probability $\kappa=\Sigma_s/\Sigma_t$=0.4. The coefficient α in the source distribution was taken as α=0.25 cm^{-1}. The detector range was taken as $|x|\le a$=2 cm. Figure 2 shows the collision density and the adjoint function, giving the expected contribution to the detector response for a particle going into a collision at x. The adjoint function exhibits jumps at $x=\pm a$ as a particle going into a collision just within the detector range will certainly contribute to the detector response, while a particle having a collision just outside the detector range will only have a probability<1 to have one or more of its next collisions in the detector. In contrast with this, the generally used adjoint function for particles *leaving* a collision or the source will not be discontinuous. With the parameters given above the detector response, chosen as the average flux over the detector range, equals R=0.14485.

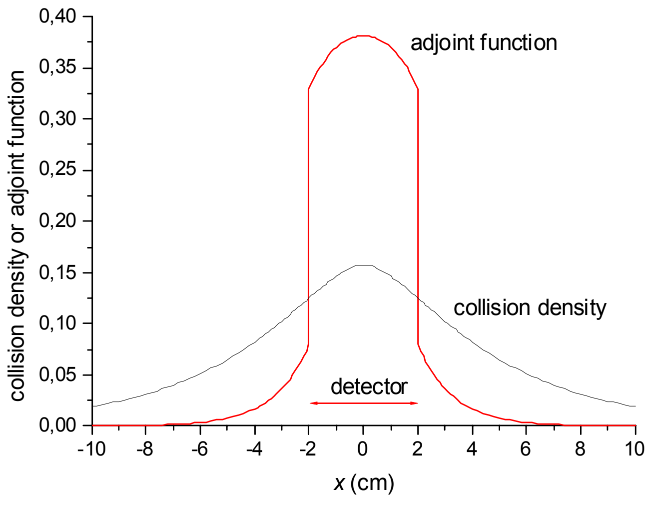

Fig. 2 Collision density and adjoint function

Figure 3 shows the result for the second moment of the score as a function of position for various values of the initial particle weight W. The threshold value for application of the Russian roulette was taken W_{th}=0.25. The particle weight after surviving the Russian roulette is W_{sv}=0.5. For large values of W the distribution converges to that without using Russian roulette as given by Eq.(39).

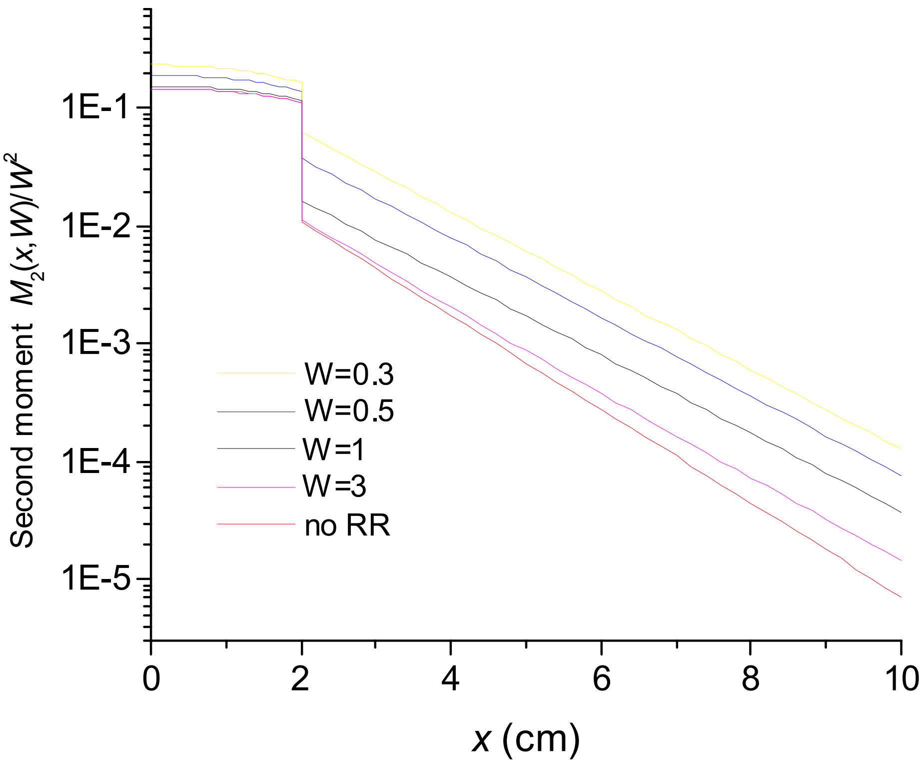

Fig. 3 Distribution of the second moment for various initial particle weights

The variance of a Monte Carlo calculation with a first-collision source $S_1(x)$ can be obtained from

$$V = \int S_1(x) \frac{M_2(x,W)}{W^2} dx - R^2. \tag{42}$$

Normally, one would choose W=1.

4.2 Number of Collisions

For the case studied here with a homogeneous medium the average number of collisions is independent of the initial position and can be calculated easily for $W = W_{sv}$: $N_c(W_{sv})$= 1/(1-κ)=1.6667. Next, it can be calculated for other initial weights. In general

$$N_c(W) = n + \frac{W}{W_{av}} \frac{\kappa^n}{1-\kappa} \quad with\ n = [\frac{\ln W_{th}/W}{\ln \kappa}] + 1.$$

Figure 4 shows the number of collisions for various initial particle weights. Discontinuities arise at W_{th}/κ^n.

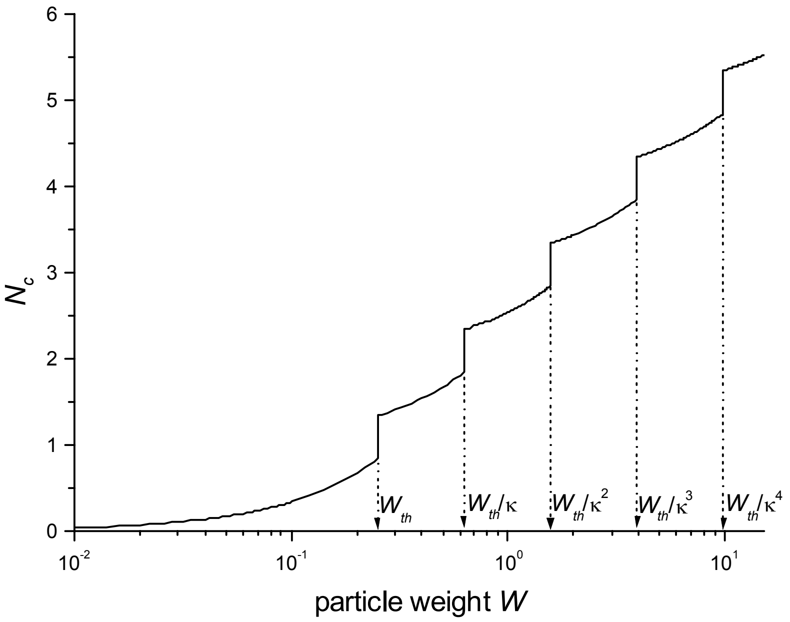

Fig. 4 Expected number of collision as a function of initial particle weight

4.3 Numerical Results

Table I shows numerical results of a Monte Carlo calculation and a comparison with theoretical results. Here the relative standard deviation is given instead of the variance for a calculation with 10^5 histories to compare directly with the Monte Carlo results. There is an excellent agreement, also in relative standard deviation and in average number of collisions, which validates the theory derived above. The last column gives the product of the variance and the number of collisions per history, the inverse of which is a measure for the efficiency. Note that the theoretical value is R=0.14485.

Table 1 Comparison of theoretical and Monte Carlo results

	Monte Carlo estimation			theoretical		
W	R	σ(%)	N_c	σ(%)	N_c	$N_c.Var$
0.3	0.1441	0.564	1.396	0.563	1.400	0.0930
0.5	0.1452	0.483	1.667	0.483	1.667	0.0816
1	0.1450	0.402	2.533	0.404	2.533	0.0867
3	0.1456	0.382	3.641	0.384	3.640	0.1123
∞	0.1452	0.380	∞	0.381	∞	∞

5. Discussion

The practical application with numerical results given in Sect. 3 and 4 shows that the theory for calculating the variance of a Monte Carlo simulation with Russian roulette is valid and can be used to obtain numerical results in simplified cases. It is clear that it will be very difficult to apply the general theory to realistic neutron or photon transport cases. Nonetheless, from the example given in Sect. 4 the usefulness of this theory for optimization of the Russian roulette can be seen. From the right most column of Table 1 one can determine the optimum value of the initial particle weight W with respect to the product of number of collisions and variance, which inverse can be seen as a measure of the efficiency of the Monte Carlo calculation. This is equivalent with optimizing the threshold value W_{th} for the application of the Russian roulette with an initial weight of unity. Of course it is also possible to optimize the particle weight W_{sv} after surviving the Russian roulette in combination with the threshold value. Moreover, the theory can be used to study the performance of different realization methods for the Russian roulette. In this paper only one way of realization was discussed. In the various general purpose Monte Carlo codes different forms are applied. The most simple form is to execute a Russian roulette with a fixed probability p, say ½, for surviving the roulette game and assigning a weight $W^*=1/p$ to the particle after survival.

As already shown by Lux and Koblinger [3] the theory can be generalized to include splitting. This opens the way to investigate the efficiency of the so-called weight window techniques often applied in general purpose Monte Carlo codes to reduce the variance and put its use on a well founded basis, which is currently not the case. [5] However, even for the simplified two-direction transport model used in this paper it is difficult to deal with Russian roulette parameters that are space dependent. A case with subdivision into two regions with regionwise constant Russian roulette parameters turns out to be feasible.

The theory in this paper is directed to a collision estimator. It is possible to extend the theory to other estimators like the track length and surface-crossing estimator.

6. Conclusion

It can be concluded that the theory derived in Sect. 2 is valid and can be actually applied to practical cases. Although application to general particle transport cases will be extremely difficult, the theory can be used to study the most efficient way to apply Russian roulette (and splitting) techniques.

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