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Convergence Analysis of 2-D/1-D Coupling Methods for the Three-Dimensional Neutron Diffusion Equation

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A convergence analysis was performed for three methods used to couple the 2-D radial and 1-D axial solutions of the 3-D neutron diffusion equation. In the first and second methods, the axial net currents and partial currents at plane interfaces were used for the inter-plane couplings, respectively. In a newly proposed third method, the current correction factors from the axial two-node kernel are used to couple the planes similar to the conventional CMFD formulation with a 2-node kernel. The new method has at least two advantages compared to the other methods. First, it is always stable whereas the net current method diverges for small mesh sizes. Second, the new method uses a Gauss-Seidel planar sweeping, and in the range of practical mesh sizes provides the best performance in terms of convergence rate.

KEYWORDS: 2-D/1-D Coupling Methods, Fourier Convergence Analysis

1. Introduction

For the last several decades, nodal methods have been used successfully for 3-D core neutronics analysis. However, because nodal methods require assembly homogenization, some drawbacks in accuracy have been observed particularly for applications with considerable heterogeneities. To remedy the problem, several efforts have been made to solve directly the heterogeneous problem with refined mesh but to reduce the computational burden by coupling 2-D planar with 1-D axial solutions using transverse leakage (TL) approximations [1,2].

However, numerical instabilities have occasionally been observed when small axial mesh sizes are used [3]. The purpose of the work here was to analyze the convergence of 2D/1D coupling methods and to investigate alternate coupling strategies that would minimize these numerical instabilities. The convergence of the three coupling methods was analyzed using Fourier analysis, which has been used successfully for convergence analysis of neutronics methods [4,5].

2. 2-D/1-D Coupling Methods

Three 2-D/1-D coupling strategies methods are investigated in this paper. They each begin with the axially averaged 2-D diffusion equation which can be written for each plane as in Eq. (1),

$$-\left(\frac{\partial}{\partial x}D_k\frac{\partial}{\partial x}+\frac{\partial}{\partial y}D_k\frac{\partial}{\partial y}\right)\overline{\phi}_k+\Sigma_k\overline{\phi}_k=\overline{Q}_k-\frac{1}{h_z}\left(J_{z,k+1}-J_{z,k}\right). \qquad (1)$$

and the radially averaged 1-D diffusion equation for each axial mesh which can be written as in Eq. (2),

$$-\frac{\partial}{\partial z} D_{i,j} \frac{\partial}{\partial z}\phi_{i,j} + \Sigma_{i,j}\phi_{i,j} = Q_{i,j} - \frac{1}{h_x}\left(J_{x,i+1} - J_{x,i}\right) - \frac{1}{h_y}\left(J_{y,j+1} - J_{y,j}\right). \tag{2}$$

The first method (method A hereafter) is to evaluate the TL of the 2-D equation directly from the axial 1-D solution. This is similar to the coupling method adopted in the DeCART code which solves the 2-D problem using MOC instead of diffusion theory [1].

The second method (method B hereafter) is to couple the 2-D/1-D equations through the partial currents. The net currents at the top and bottom of each plane in the TL of the 2-D equation can be split into incoming and outgoing partial currents. The outgoing partial currents can be expressed in terms of incoming partial currents and node average fluxes by applying a nodal method to the 1-D equation. The node average flux term in the TL of the 2-D equation can be moved to the left hand side of the equation. D. Lee et al. used this kind of method in their work [2].

The third method (method C hereafter) is to couple the 2-D/1-D equations through a current correction factor (CCF) and the average fluxes of the lower and upper planes. The CCF is widely used to preserve the net current of a higher order local solution in a lower order global finite difference solution [6]. The currents in the CCF method are written as in Eq. (3).

$$J_{z,k} = -\tilde{D}_{z,k}\left(\overline{\phi}_k - \overline{\phi}_{k-1}\right) + \hat{D}_{z,k}\left(\overline{\phi}_k + \overline{\phi}_{k-1}\right). \tag{3}$$

By inserting Eq. (3) into Eq. (1) and moving the term to the left hand side, we can obtain

$$-\left(\frac{\partial}{\partial x} D_k \frac{\partial}{\partial x} + \frac{\partial}{\partial y} D_k \frac{\partial}{\partial y}\right)\overline{\phi}_k + \tilde{\Sigma}_k \overline{\phi}_k = \overline{Q}_k + S_k\,, \tag{4}$$

where

$$\tilde{\Sigma}_k = \Sigma_k + \frac{\tilde{D}_{z,k+1} + \hat{D}_{z,k+1} + \tilde{D}_{z,k} - \hat{D}_{z,k}}{h_z}\,,$$

$$S_k = \frac{1}{h_z}\left[\left(\tilde{D}_{z,k+1} - \hat{D}_{z,k+1}\right)\overline{\phi}_{k+1} + \left(\tilde{D}_{z,k} + \hat{D}_{z,k}\right)\overline{\phi}_{k-1}\right].$$

This strategy is the new method proposed in this paper.

3. Convergence Analysis

A model problem was introduced to analyze the convergence rate of the coupling strategies. The model problem here is a three dimensional one-group diffusion problem with a flat fixed source in a homogeneous non-multiplying infinite medium. Two basic assumptions are introduced to analyze the convergence rate of the strategies for the model problem. These are (1) solving the 2-D problems plane by plane, which means solving iteratively in the z-direction and (2) solving the 2-D problem by direct inversion of the 2-D operator in a given plane. The second assumption leads to a zero radial leakage during the iterations which leads to the following 1-D/2-D equations.

$$\Sigma\bar{\phi}_k = Q - \frac{1}{h}\left(J_{z,k+1} - J_{z,k}\right), \tag{5}$$

$$-D\frac{\partial^2}{\partial z^2}\phi + \Sigma\phi = Q, \tag{6}$$

3.1 Fourier Analysis of Method A

The iteration algorithm of method A can be expressed by the following equations :

$$\Sigma h\bar{\phi}_k^{(n)} = Qh - \left(J_{z,k+1}^{(n-1)} - J_{z,k}^{(n-1)}\right), \tag{7}$$

$$J_{z,k}^{(n-1)} = -A\left(\bar{\phi}_k^{(n-1)} - \bar{\phi}_{k-1}^{(n-1)}\right), \tag{8}$$

where

$$A = \frac{\Sigma h}{4\sinh^2\left[h/(2L)\right]}.$$

The two-node analytic nodal method was used to solve the 1-D equation

A first order perturbation of flux from the exact solution at iteration step n is introduced as follows:

$$\bar{\phi}_k^{(n)} = \frac{Q}{\Sigma}\left(1 + \varepsilon\xi_k^{(n)}\right). \tag{9}$$

From Eq. (7), Eq. (8), and Eq. (9), the following equation is obtained:

$$\Sigma h\xi_k^{(n)} = A\left(\xi_{k-1}^{(n-1)} - 2\xi_k^{(n-1)} + \xi_{k+1}^{(n-1)}\right). \tag{10}$$

By inserting the following Fourier ansatz into Eq. (10)

$$\xi_k^{(n)} = a\omega^n e^{i\lambda(k+1/2)h}, \tag{11}$$

it is possible to obtain the following convergence rate :

$$\omega = \frac{\cos(\tau) - 1}{2\sinh^2\left[h/(2L)\right]}; \quad \tau = \lambda h \tag{12}$$

with a spectral radius given by:

$$\rho = \sup_{\tau\in[0,2\pi]}|\omega| = \frac{1}{\sinh^2[h/(2L)]} \quad . \tag{13}$$

3.2 Fourier Analysis of Method B

The iterative algorithm of method B can be expressed by the following equations :

$$(\Sigma h + 2\gamma)\overline{\phi}_k^{(n)} = Qh - (\alpha+\beta-1)\left(j_{z,k}^{+(n)} + j_{z,k+1}^{-(n-1)}\right), \tag{14}$$

$$j_{z,k}^{-(n)} = \alpha\, j_{z,k}^{+(n)} + \beta\, j_{z,k+1}^{-(n-1)} + \gamma\,\overline{\phi}_k^{(n)}, \tag{15}$$

$$j_{z,k+1}^{+(n)} = \beta\, j_{z,k}^{+(n)} + \alpha\, j_{z,k+1}^{-(n-1)} + \gamma\,\overline{\phi}_k^{(n)}, \tag{16}$$

where α , β , and γ are functions of D , Σ , and h .

A first order perturbation for the average flux, Eq. (9), and the following expression for the partial currents are obtained:

$$j_{z,k}^{\pm(n)} = \frac{Q}{4\Sigma}\left(1 + \varepsilon\eta_k^{\pm(n)}\right), \tag{17}$$

which can be introduced to obtain the following equations :

$$(\Sigma h + 2\gamma)\xi_k^{(n)} = -\frac{\alpha+\beta-1}{4}\left(\eta_k^{+(n)} + \eta_k^{-(n-1)}\right), \tag{18}$$

$$\eta_k^{-(n)} = \alpha\,\eta_k^{+(n)} + \beta\,\eta_{k+1}^{-(n-1)} + 4\gamma\,\xi_k^{(n)}, \tag{19}$$

$$\eta_{k+1}^{+(n)} = \beta\,\eta_k^{+(n)} + \alpha\,\eta_{k+1}^{-(n-1)} + 4\gamma\,\xi_k^{(n)}, \tag{20}$$

By inserting Eq. (11) and the following Fourier ansatz into Eq. (18), Eq. (19), and Eq. (20)

$$\eta_k^{\pm(n)} = b^{\pm}\omega^n e^{i\lambda kh}, \tag{21}$$

the convergence rate, ω , and the spectral radius, ρ , for method B are obtained.

3.3 Fourier Analysis of Method C

The iteration algorithm of method C can be expressed by the following equations :

$$\left(\Sigma h + 2D + \hat{D}_{z,k+1}^{(n)} - \hat{D}_{z,k}^{(n)}\right)\overline{\phi}_k^{(n)} = Qh + \left(D + \hat{D}_{z,k}^{(n)}\right)\overline{\phi}_{k-1}^{(n)} + \left(D - \hat{D}_{z,k+1}^{(n)}\right)\overline{\phi}_{k+1}^{(n-1)}, \tag{22}$$

$$\hat{D}_{z,k}^{(n)} = \frac{(D-A)\left(\bar{\phi}_k^{(n-1)} - \bar{\phi}_{k-1}^{(n-1)}\right)}{\bar{\phi}_k^{(n-1)} + \bar{\phi}_{k-1}^{(n-1)}}. \tag{23}$$

Inserting the first order perturbation for average flux, Eq. (9), and using the following expression for the CCF:

$$\hat{D}_{z,k}^{(n)} = \varepsilon \tau_k^{(n)} \tag{24}$$

and dropping $O\left(\varepsilon^2\right)$ nonlinear terms yields the following linearized equations :

$$(\Sigma h + 2D)\xi_k^{(n)} = 2\left(\tau_k^{(n)} - \tau_{k+1}^{(n)}\right) + D\left(\xi_{k-1}^{(n)} + \xi_{k+1}^{(n-1)}\right), \tag{25}$$

$$2\tau_k^{(n)} = (D-A)\left(\xi_k^{(n-1)} - \xi_{k-1}^{(n-1)}\right). \tag{26}$$

By inserting Eq. (11) and the following Fourier ansatz into Eq. (25) and Eq. (26)

$$\tau_k^{(n)} = c\omega^n e^{i\lambda kh}, \tag{27}$$

the convergence rate, ω, and the spectral radius, ρ, for method C can be obtained.

4. Results and Conclusions

The spectral radii of each method is shown in Figure 1 for the model problem with $D = 0.83333$ and $\Sigma = 0.02$. The lines in Figure 1 are the analytic spectral radius obtained by Fourier analysis and the dots are the numerically evaluated ones. Good agreement is observed between the analytic and numerical results.

It was noted that the spectral radius of method A approaches infinity as the mesh size goes to zero, which indicates that the solution will diverge as the mesh size become small. The same behavior was observed with numerical tests in the DeCART code [3]. Conversely, the spectral radii of strategies B and C are always less than 1, which indicates that the algorithms are always stable. As indicated, Method C shows better performance than method B.

In conclusion, a new 2-D/1-D coupling method based on current correction factors was proposed in this paper and the convergence behavior of three 2-D/1-D coupling strategies were analyzed using Fourier analysis. It was found that the newly proposed method provides the best overall performance in terms of stability and rate of convergence.

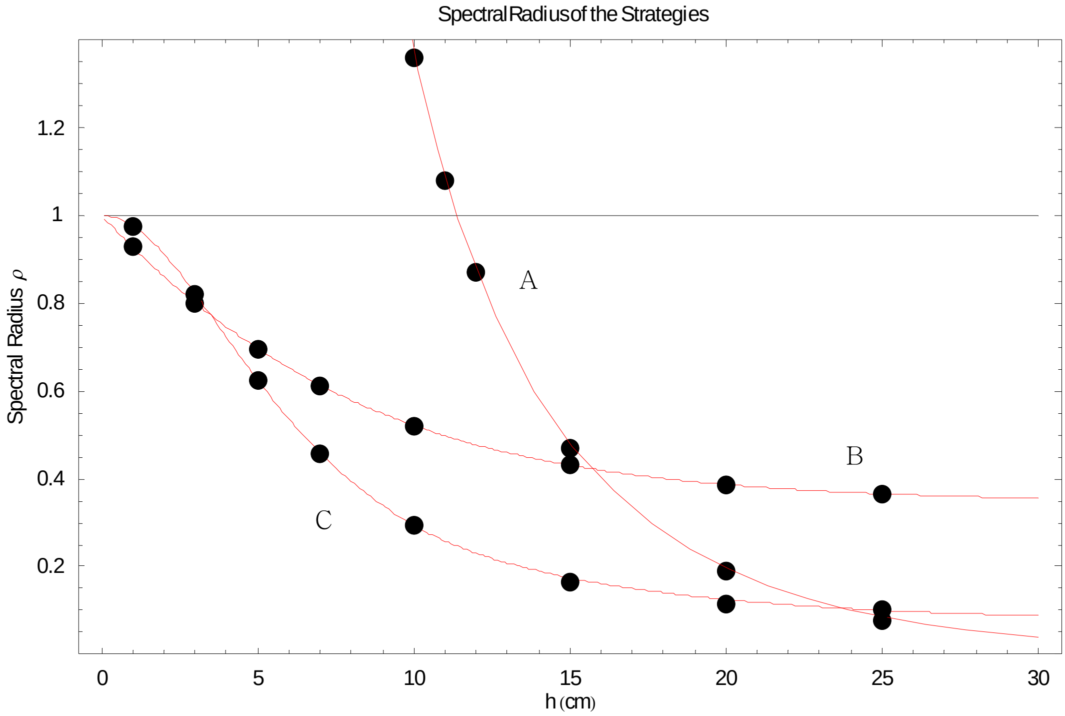

Fig. 1. Spectral Radii of the Coupling Methods

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