

## Lambda modes of the neutron diffusion equation in hexagonal geometry

Teresa Barrachina\*<sup>1</sup> Damián Ginestar<sup>2</sup> and Gumersindo Verdú<sup>1</sup>

<sup>1</sup>Departamento de Ingeniería Química y Nuclear. Universidad Politécnica de Valencia  
 Camino de Vera, 14. 46022 Valencia. Spain

<sup>2</sup> Departamento de Matemática Aplicada. Universidad Politécnica de Valencia Camino de  
 Vera, 14. 46022 Valencia. Spain

### Abstract

A nodal collocation method is proposed to compute the dominant Lambda modes of nuclear reactor core with a hexagonal geometry. This method is based on a triangular mesh and assumes that the neutronic flux can be approximated as a finite expansion in terms of Dubiner's polynomials. The method transforms the initial differential eigenvalue problem into a generalized algebraic one, from which the dominant modes of the reactor can be computed. The performance of the method is tested with two benchmark problems.

**KEYWORDS:** *Lambda modes, hexagonal geometry, neutron diffusion equation nodal collocation method*

### 1. Introduction

For a given configuration of a nuclear power reactor core, it is always possible to force its criticality dividing the neutron production rate due to fission by a positive number,  $\lambda$ , obtaining a neutron balance equation of the form [1]

$$L \phi = \frac{1}{\lambda} M \phi, \quad (1)$$

where  $L$  is the neutron loss operator and  $M$  is the neutron production operator. Equation (1) is a generalized eigenvalue problem known as the Lambda modes equation. The largest eigenvalue  $\lambda$  satisfying (1) is the *k-effective* of the reactor core, and its corresponding eigenfunction gives the neutronic flux of a stationary configuration of the core. We will consider the approximation of two groups of energy, that the neutrons are born in the fast group, and that there is not up-scattering from the thermal to the fast group. Taking into account these assumptions, the losses operator is

$$L = \begin{bmatrix} -\bar{\nabla}(D_1 \bar{\nabla}) + \Sigma_{a1} + \Sigma_{12} & 0 \\ -\Sigma_{12} & -\bar{\nabla}(D_2 \bar{\nabla}) + \Sigma_{a2} \end{bmatrix}.$$

The production operator and the neutron flux are

$$M = \begin{bmatrix} \nu \Sigma_{f1} & \nu \Sigma_{f2} \\ 0 & 0 \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}.$$

As a first approximation, we consider zero-flux boundary conditions.

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\* Corresponding author, Tel. 34-963878669, Fax. 34-963877669, E-mail: [tebarcel@mat.upv.es](mailto:tebarcel@mat.upv.es)

The Lambda modes have been successfully used to develop modal methods to solve the time dependent neutron diffusion equation [2], which can be viewed as generalizations of the quasistatic method [3]. These methods are fast and accurate when the spatial power distribution changes slowly during the transient. This makes interesting to obtain efficient method to compute the Lambda modes of a reactor core.

The dominant Lambda modes of a nuclear power reactor core have been obtained for cores with rectangular geometry as PWR and BWR reactors (see [4] and references therein). In this paper, we present a method to discretize the Lambda modes equation in 2D geometry, when the nodes or cells used for the discretization are triangles instead of rectangles. This method can be used to compute the dominant Lambda modes of reactors with hexagonal geometry as the VVER type reactors, dividing the natural hexagonal cells into six equilateral triangles.

Different methods have been proposed to solve the neutron diffusion equation in hexagonal geometry as, for example, the Fourier transform method [5], the conformal mapping method [6], the polynomial expansion nodal method [7], etc. These methods focus on the determination of the *k-effective* and the stationary neutronic flux distribution in the reactor core. To be able to determine the dominant modes of a reactor core with hexagonal geometry, we use an orthogonal nodal collocation method for triangles that allows approximating the initial differential eigenvalue problem (1) by a generalized algebraic eigenvalue problem from which the dominant Lambda modes can be computed, using an efficient numerical procedure.

The rest of the paper is organized as follows, in section 2, the nodal collocation method used to discretize the Lambda modes problem on a triangular mesh is presented. To test the performance of the method, a homogeneous reactor and the benchmark problem IAEA reactor [8] are studied in section 3. The main conclusions of the paper are summarized in section 4.

## 2. Nodal collocation method

For the development of the nodal collocation method it is assumed that the neutronic flux in each one of the triangles of the discretization,  $e$ , can be expanded in terms of the Dubiner polynomials [9]

$$\phi_e(x', y') = \sum_{m,n=0}^{m+n \leq K} \phi_{e,mn} g_{mn}(x', y'). \quad (2)$$

The Dubiner's polynomials,  $g_{mn}(x, y)$ , satisfy the orthogonality relation

$$\iint_T g_{mn}(x', y') g_{l's'}(x', y') dx' dy' = N_{mn} \delta_{ml} \delta_{ns},$$

where  $T$  is the right triangle, defined as

$$T = \{(x', y') / x' \geq 0, y' \geq 0, 0 \leq x' + y' \leq 1\}.$$

To develop the nodal collocation method we start from the Lambda modes equation in the approximation of one group of energy (the generalization to multigroup approximation is straightforward),

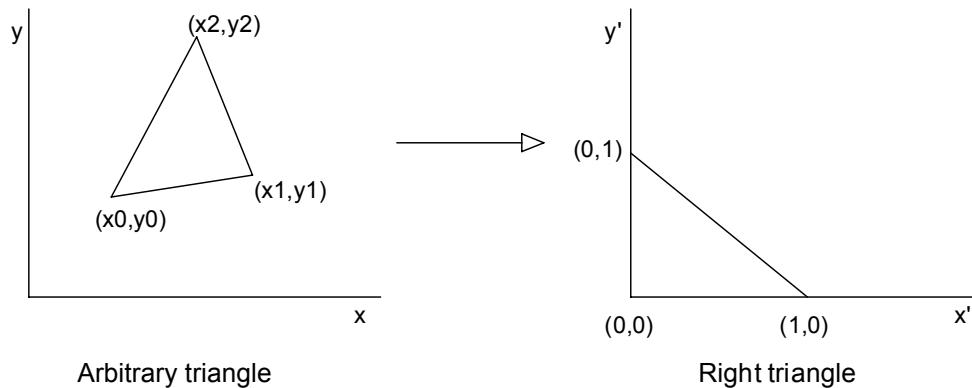
$$-D_x \frac{\partial^2 \phi}{\partial x^2} - D_y \frac{\partial^2 \phi}{\partial y^2} + \Sigma_a \phi = \frac{1}{\lambda} \nu \Sigma_f \phi, \quad (3)$$

and by means of an affine transformation, the equation (3) for each triangle,  $e$ , of the mesh, is transformed onto the right triangle (see Fig. 1), obtaining an equation of the form

$$-\bar{D}_{x'} \frac{\partial^2 \phi_e}{\partial x'^2} - \bar{D}_{x'y'} \frac{\partial^2 \phi_e}{\partial x' y'} - \bar{D}_{y'} \frac{\partial^2 \phi_e}{\partial y'^2} + \Sigma_a \phi_e = \frac{1}{\lambda} \nu \Sigma_f \phi_e, \quad e = 1, \dots, N, \quad (4)$$

where  $N$  is the number of triangles used in the discretization of the reactor core.

**Figure 1:** Transformation from an arbitrary triangle onto the right triangle.



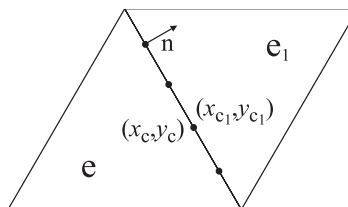
A first set of equations is obtained for all the triangles of the discretization from the integral relations

$$\iint_T g_{ij}(x', y') \left( -\bar{D}_{x'} \frac{\partial^2 \phi_e}{\partial x'^2} - \bar{D}_{x'y'} \frac{\partial^2 \phi_e}{\partial x' y'} - \bar{D}_{y'} \frac{\partial^2 \phi_e}{\partial y'^2} + \Sigma_a \phi_e \right) dx' dy' = \frac{1}{\lambda} \nu \Sigma_f \iint_T g_{ij}(x', y') \phi_e dx' dy', \quad (5)$$

with  $i, j = 0, \dots, i + j \leq K - 3, e = 1, \dots, N$ .

We have  $N(K+1)(K+2)/2$  unknowns from the expansions, (2). The remaining  $3KN$  equations necessary to determine the system, are obtained imposing continuity conditions for the neutronic flux and the neutronic current normal to each face on a set of collocation points, known as the Fekete points of the right triangle [10]. These collocation points are placed on the internal edges of the triangles of the mesh (see Fig. 2).

**Figure 2:** Continuity conditions for the neutronic flux and current.



Special continuity conditions for the current must be satisfied in the interior vertex of the triangular mesh. With these continuity conditions, we obtain a set of equations that approximately guarantee the neutron balance among the different nodes of the mesh. Finally, boundary conditions are imposed on the Fekete points placed on the external edges of the mesh.

With the nodal collocation method we can approximate the two energy groups approximation of the Lambda modes problem (1) by a generalized algebraic eigenvalue problem with the following block structure

$$\begin{bmatrix} L_{11} & 0 \\ -L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} M_{11} & M_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \quad (6)$$

To solve this problem, it is reduced to the ordinary eigenvalue problem

$$L_{11}^{-1} (M_{11} + M_{12} L_{22}^{-1} L_{21}) \psi_1 = \lambda \psi_1, \quad (7)$$

which is solved for the dominant eigenvalues and their corresponding eigenvectors using the Implicit Restarted Arnoldi Method [11].

### 3. Numerical results

#### 3.1 Homogeneous rectangle

The Lambda modes problem for a homogeneous rectangle with sides lengths  $L_1$  and  $L_2$ , using zero-flux boundary conditions, has an analytical solution. The eigenvalues solutions for this problem are of the form

$$\lambda = \frac{\nu \Sigma_{f1} (D_2 B_{m,n}^2 + \Sigma_{a2}) + \nu \Sigma_{f2} \Sigma_{12}}{D_1 (D_2 (B_{m,n}^2)^2 + \Sigma_{a2} B_{m,n}^2) + (\Sigma_{a1} + \Sigma_{a2}) (D_2 B_{m,n}^2 + \Sigma_{a2})}, \quad (8)$$

where

$$B_{m,n}^2 = \left( \frac{m\pi}{L_1} \right)^2 + \left( \frac{n\pi}{L_2} \right)^2, \quad m, n = 1, 2, \dots$$

For this problem, we have considered  $L_1=L_2=100$  cm and the nuclear cross sections given in Tab. 1. The results for the four dominant eigenvalues calculated with expression (8) are presented in Tab. 2.

**Table 1:** Nuclear cross sections for the homogeneous rectangle.

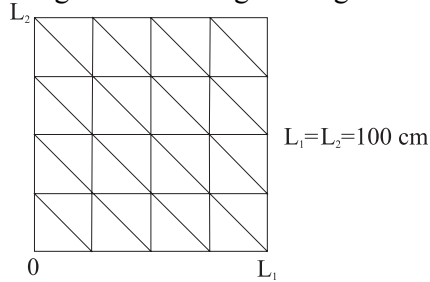
$D_1$	$D_2$	$\Sigma_{a1}$	$\Sigma_{a2}$	$\Sigma_{12}$
1.3466	0.37169	$0.8362 \cdot 10^{-2}$	$6.4277 \cdot 10^{-2}$	$1.6893 \cdot 10^{-2}$
$\nu \Sigma_{f1}$	$\nu \Sigma_{f2}$			
$4.4488 \cdot 10^{-3}$	$7.3753 \cdot 10^{-2}$			

**Table 2:** Analytical dominant eigenvalues for the homogeneous rectangle.

$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$
0.845965	0.730229	0.730229	0.640501

To use the nodal collocation method exposed above, we have considered the triangular mesh presented in Fig. 3. The results obtained for the first four dominant eigenvalues computed using different values of the number of polynomials,  $K$ , considered in the nodal collocation method are presented in Tab. 3.

**Figure 3:** Homogeneous rectangle triangular nodalization



**Table 3:** Eigenvalues of the homogeneous rectangle in terms of  $K$ .

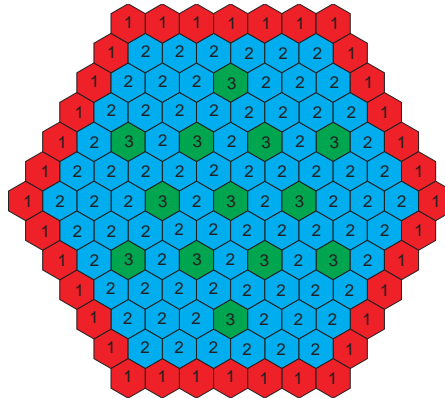
Degree ( $K$ )	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$
3	0.84093	0.71992	0.70750	0.60381
4	1.34489	0.84585	0.72801	0.71550
5	0.84596	0.73022	0.73021	0.64039
6	0.84596	0.73023	0.73014	0.64036
7	0.84596	0.73023	0.73023	0.64050

We observe that when  $K=7$ , the nodal collocation method converges to the analytical results.

### 3.2 2D IAEA reactor

The IAEA reactor is an International Atomic Agency (IAEA) benchmark [6]. The core is a little smaller than the commercial VVER-1000 reactor. There are 13 rodged assemblies in the core. The discretization in hexagons of the core and the materials distribution are shown in Fig. 4.

**Figure 4:** IAEA reactor without reflector.



Tab. 4 shows the nuclear parameter data for the IAEA core. The fuel assembly pitch is 20 cm and the boundary conditions considered are zero-flux conditions.

**Table 4:** Nuclear cross sections for the IAEA core.

Mat.	$D_1$	$D_2$	$\Sigma_{a1}$	$\Sigma_{a2}$	$\Sigma_{12}$	$\nu\Sigma_{f1}$	$\nu\Sigma_{f2}$
1	1.5	0.4	0.01	0.080	0.02	0.0	0.135
2	1.5	0.4	0.01	0.085	0.02	0.0	0.135
3	1.5	0.4	0.01	0.13	0.02	0.0	0.135

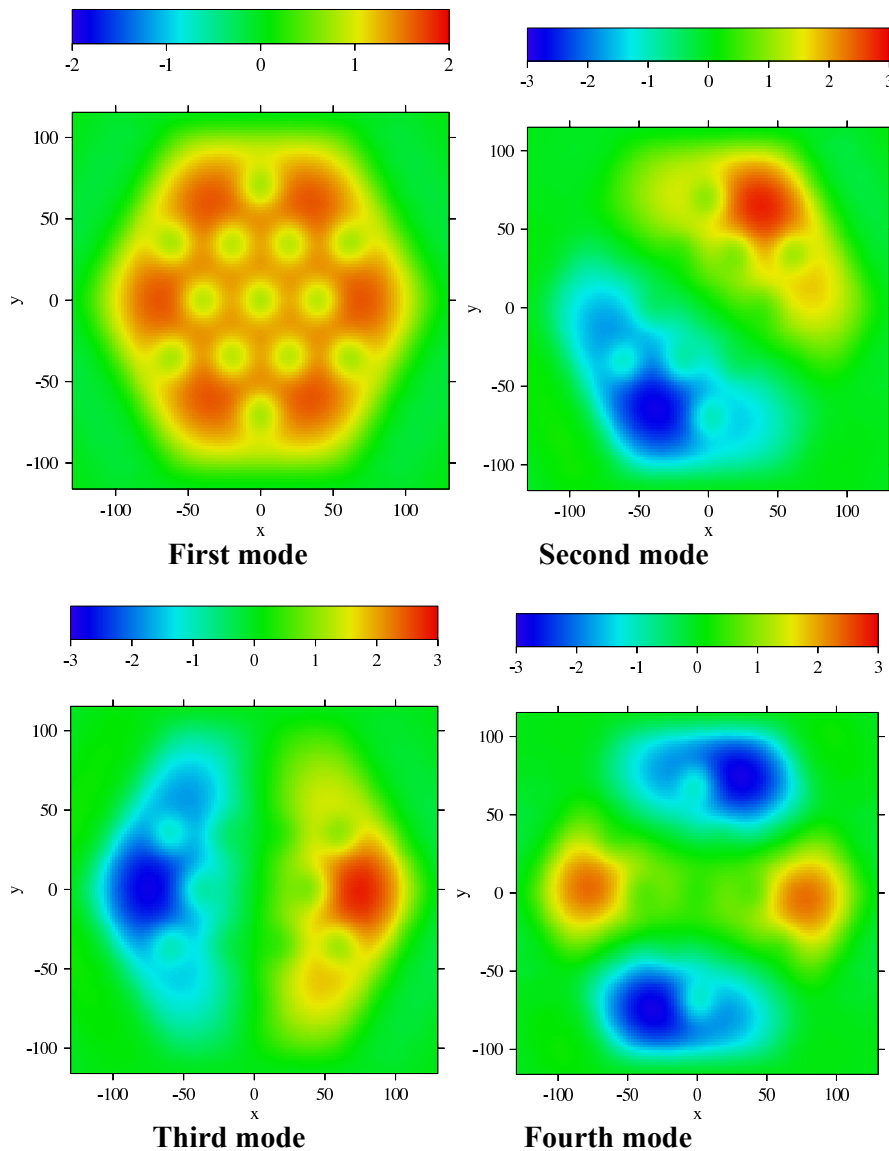
To test the performance of the nodal collocation method, we have calculated the *k-effective* of the reactor core using the neutronic module PARCS [7], obtaining a value of  $\lambda_1 = 0.974545$ . In Tab. 5 we show the obtained results for the first four eigenvalues of the core using different values for the number of polynomials  $K$ .

**Table 5:** Eigenvalues for the IAEA reactor core for different values of  $K$ .

Degree ( $K$ )	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$
4	0.98853	0.98810	0.98810	0.97594
5	0.97386	0.95745	0.93234	0.91438
6	0.97604	0.95524	0.95524	0.92998
7	0.97414	0.95748	0.95748	0.93144
8	0.97295	0.95630	0.95630	0.93016

The relative power distribution associated with each one of the modes is shown in Fig 5.

**Figure 5:** Power distribution associated with the four dominant modes of the IAEA reactor core.



From Fig. 5 we observe that the different eigenfunctions present different symmetry patterns and for their calculation it is necessary to consider the core as a whole. In this way it is not possible to make use of the 1/12 symmetry that presents the core with respect to the materials distribution.

#### 4. Conclusions

In this paper we have presented a nodal collocation method for the computation of the dominant Lambda modes of a given configuration of a nuclear reactor core using a triangular mesh. This method is based on the expansion of the neutronic flux in terms of orthogonal polynomials on the right triangle. The neutron balance conditions for each triangle of the mesh are imposed by means of integral conditions on the node. To assure the conservation of the neutron balance among the different nodes continuity conditions for the neutronic flux and

current are imposed on a set of collocation points on the internal edges of each triangle.

The developed method succeeds in computing the dominant modes for 2D reactors as the benchmark IAEA reactor core without reflector.

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