Two-Group CMFD Accelerated Multi-group Calculation with a Semi-Analytic Nodal Kernel

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Abstract

An efficient multigroup nodal calculation method based on the two-group CMFD formulation and a multigroup one-node semi-analytic nodal method (SANM) kernel is presented. The SANM solution here is based on a quartic expansion of all the source terms. The three transverse integrated one-dimensional equations for a three-dimensional one-node problem specified with the incoming current boundary conditions are solved simultaneously for the outgoing currents and the node average flux. With the use of the two-group CMFD structure for the global calculation, the multigroup calculation needs to be performed only at the local one-node level and thus the implementation of this method into an existing code is facilitated. The effectiveness of this method is examined by solving a set of multigroup problems. The results of a set of sensitivity studies to determine the number of one nodes sweeps per nodal update, order of the energy and space sweeps and group split pattern are also presented.

KEYWORDS: SANM, CMFD, Multigroup Nodal Method, One-Node kernel

1. Introduction

The coarse mesh finite difference (CMFD) formulation or the nonlinear nodal method [1] has been widely used for the solution of the two-group (2-G) neutron diffusion problems. Because of its high efficiency and simplicity, the 2-G CMFD formulation is employed in numerous nodal diffusion codes for light water reactors (LWRs). As the core design becomes more innovated such as the MOX loaded cores or the graphite moderated high temperature cores, there is more need for an efficient multigroup solver which would encompass the transient calculation as well. This paper is to introduce an efficient multigroup solution method within the framework of the 2-D CMFD formulation such that the implementation into an existing code is facilitated. This method employs a one-node formulation of the semi-analytic nodal method (SANM) [2] that can be easily extended to the multi-group calculation. With the use of the two-group structure for the global calculation, the multigroup calculation needs to be performed only at the local one-node level.

There have been many variations of the SANM. Esser and Smith[3] introduced the

semianalytic two-group nodal method which keeps the analytic solution only for the thermal group flux by introducing a 4-th order polynomial expansion for the fast group flux. Kim et al. introduced the SANEM method which involves the analytic solution obtained after moving all the source terms to the right hand side and then approximating them with a 4-th polynomial for each group. The analytic solution consists of two exponential function terms and a 4-th order polynomial. This method is readily applied to multigroup problems while Esser's formulation is limited to two-groups. Fu and Cho [4] employed similar approach as Kim, but the transverse leakage term is limited to quadratic. They, however, examined the CMFD formulation with the two-node solution kernel. In this work, a one-node based SANM kernel is derived by introducing the simultaneous solution of the three directional outgoing fluxes and the node average flux of a node with the same form of the intra-ndal flux function as SANEM. The one node multigroup SANM kernel is then implemented within the framework of the two-group CMFD formulation involving dynamic group condensation [5]. In order to examine the accuracy and efficiency of this method, several multigroup eigenvalue problems are investigated with a set of fast reactor cross sections.

2. Method

2.1 One-Node Semi-Analytic Kernel

With the source iteration scheme in which all the sources consisting of the fission, scattering, transverse leakage, and possible transient fixed sources is placed on the right hand side of the transverse integrated neutron balance equation, the SANM can be readily applied to the solution of multi-group problems once the spatial distribution of the source term is approximated by a high order polynomial. For a one-node problem, a new SANM formulation is derived below with an emphasis on the simultaneous solution of the three directional solutions.

The transverse integration of the following multi group diffusion equation

$$-D_g^m \nabla^2 \phi_g^m(\mathbf{r}) + \Sigma_{tg}^m \phi_g^m(\mathbf{r}) = \frac{\chi_g}{k_{eff}} \sum_{g'=1}^{G} \nu\Sigma_{fg'}^m \phi_{g'}^m(\mathbf{r}) + \sum_{g'=1}^{G} \Sigma_{sgg'}^m \phi_{g'}^m(\mathbf{r}) + S_g^m(\mathbf{r}) \tag{1}$$

yields the following one-dimensional multi-group diffusion equation:

$$-\mathbf{D}\frac{d^2}{du^2}\phi_u(u) + \mathbf{R}\phi_u(u) = \frac{1}{k_{eff}}\mathbf{F}\phi_u(u) + \mathbf{C}\phi_u(u) + \mathbf{S}_u(u) - \mathbf{L}_u(u) \quad (u = x, y, z) \tag{2}$$

$$\mathbf{R} = \begin{bmatrix} \Sigma_{r1} & & 0 \\ & \ddots & \\ 0 & & \Sigma_{rG} \end{bmatrix}, \mathbf{F} = \begin{bmatrix} \chi_1\Sigma_{f1} & \cdots & \chi_1\Sigma_{fG} \\ \vdots & \ddots & \vdots \\ \chi_G\Sigma_{f1} & \cdots & \chi_G\Sigma_{fG} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & & 0 \\ \Sigma_{sg1} & \ddots & \\ \Sigma_{sG1} & \Sigma_{sGg} & 0 \end{bmatrix}$$

Here, the right hand side, defined as the effective source term, shall be approximated by a quadratic polynomial as the following form:

$$\tilde{Q}(u) = \sum_{i=0}^{4} \tilde{Q}_{iug} g_{iu}\left(u/a_u\right) + \sum_{i=0}^{2} S_{iug} g_{iu}\left(u/a_u\right) \tag{3}$$

$$g_{0u}(\tau) = 1,\ g_{1u}(\tau) = 2\tau,\ g_{2u}(\tau) = 6\tau^2 - 1/2,$$

$$g_{3u}(\tau) = 2\tau\left(\tau - 1/2\right)\left(\tau + 1/2\right),\ g_{4u}(\tau) = \left(5\tau^2 - 1/4\right)\left(\tau - 1/2\right)\left(\tau + 1/2\right)$$

In Eq. (3) the second term represents the quadratic transverse leakage. Given this approximation, one can obtain the analytic solution Eq.(2) which consists of the homogeneous and particular solutions of the following form:

$$\phi_{ug}(u) = \sum_{i=0}^{6} C_{iug} g_{iu}\left(u/a_u\right) \tag{4}$$

$$g_{5u}(\tau) = a_5 \sinh(2\tilde{\kappa}_g \tau) + b_5 \tau,\ g_{6u}(\tau) = a_6 \cosh(2\tilde{\kappa}_g \tau) + b_6 \tau^2 + c_6,$$

$$\tilde{\kappa}_g = \frac{a_u \sqrt{\Sigma_{rg}/D_g}}{2}$$

The first five basis functions constitute the particular solution of Eq.(2) and remaining two form the homogeneous solution of Eq.(2).

The basis functions of the particular solution that we used here are constructed from linear combinations of five independent polynomial functions. Note also that those functions the same ones that have been used in the NEM formulation [6]. The two basis functions for the homogeneous solution, $g_i(\tau)$ $(i = 5,6)$, are determined so that they satisfy the following conditions:

$$g_{iu}\left(\pm\frac{1}{2}\right) = 0 \quad (i = 5,6),\ \int_{-1/2}^{+1/2} g_{iu}(\tau)\, d\tau = 0 \quad (i = 5,6), \tag{5}$$

$$g'_{5u}\left(\pm\frac{1}{2}\right) = +1,\ g'_{6u}\left(\pm\frac{1}{2}\right) = \pm 1$$

Note that the above conditions are met as by the polynomial basis functions of the NEM.

The coefficients of the one-dimensional flux, C_{iug} except the constant term, can be obtained from the given right hand side function, Eq. (3), and the incoming current boundary

condition given at the two end. In principle, the constant term should also be determined as long as the constant term in Eq. (3) is fixed. However, the constant term consists of the average transverse leakage which is to be determined simultaneously form the solution for the other directions. Thus the constant term in Eq. (3) should be treated as an unknown. Consequently, the constant term in the solution, Eq. (4), can not be determined. However, it is possible to introduce the node average flux as the unknown to be determined and to impose the condition that the average of Eq. (4) be the node average flux. Because the integral of all the basis function is null by the condition given by Eq. (5), the constant term in the solution is merely the node average flux which is to be determined later.

Once the coefficients are determined, the following relation for the outgoing partial current is obtained:

$$j^{+}_{urg} = T_{0ug}\left(6\bar{\phi} - C_{4ug} - C_{6ug}\right) - T_{1ug}\left(C_{3ug} + C_{5ug}\right) - T_{2ug} j^{+}_{ulg} + T_{3ug} j^{-}_{urg} \tag{7}$$

$$j^{-}_{ulg} = T_{0ug}\left(6\bar{\phi} - C_{4ug} - C_{6ug}\right) + T_{1ug}\left(C_{3ug} + C_{5ug}\right) - T_{2ug} j^{-}_{urg} + T_{3ug} j^{+}_{ulg}$$

with T_{iug} is a constant given in terms of $\beta_{ug} = D_g / a_u$.

Here the unknown node average flux is coupled with the outgoing currents through the whole nodal balance equation, from which the node average flux can be obtained as:

$$\bar{\phi}_g = \frac{\tilde{H}_g + \sum_{u=x,y,z} \frac{1}{a_u}\left(2T_{0ug}(C_{4ug} + C_{6ug}) + (1 + T_{2ug} - T_{3ug})(j^{+}_{ulg} + j^{-}_{urg})\right)}{\Sigma_{rg} + 12 \sum_{u=x,y,z} T_{0ug} / a_u} \tag{8}$$

$$\tilde{H}_g = \frac{\chi_g}{k_{eff}} \sum_{g'=1}^{G} \nu\Sigma_{fg'}\bar{\phi}_{g'} + \sum_{\substack{g'=1 \\ g' \neq g}}^{G} \Sigma_{sgg'}\bar{\phi}_{g'}$$

In the conventional procedure, the outgoing currents are determined first using the old value of average flux and the average flux is then updated using the new outgoing currents. Here, however, a simultaneous solution is obtained for the outgoing currents and the node average flux by solving the which yields the following coupled linear equation:

$$j^{+}_{urg} - T_{0u}\bar{\phi} = T_{1u}C_{1ug} + T_{2u}C_{2ug} + T_{3u}C_{3ug} + T_{4u}C_{4ug} + T_{5u} j^{+}_{ulg} + T_{6u} j^{-}_{urg} \tag{9}$$

$$j^{-}_{ulg} - T_{0u}\bar{\phi} = -T_{1u}C_{1ug} + T_{2u}C_{2ug} - T_{3u}C_{3ug} + T_{4u}C_{4ug} + T_{5u} j^{-}_{urg} + T_{6u} j^{+}_{ulg}$$

$$\sum_{u=x,y,z} \frac{1}{a_u}\left(j^{+}_{urg} + j^{-}_{ulg}\right) + \Sigma_{rg}\bar{\phi}_g = \tilde{H}_g + \sum_{u=x,y,z} \frac{1}{a_u}\left(j^{-}_{urg} + j^{+}_{ulg}\right)$$

These equations can be represented more concisely using a matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & a_{17} \\ 0 & 1 & 0 & 0 & 0 & 0 & a_{27} \\ 0 & 0 & 1 & 0 & 0 & 0 & a_{37} \\ 0 & 0 & 0 & 1 & 0 & 0 & a_{47} \\ 0 & 0 & 0 & 0 & 1 & 0 & a_{57} \\ 0 & 0 & 0 & 0 & 0 & 1 & a_{67} \\ a_{71} & a_{72} & a_{72} & a_{72} & a_{72} & a_{72} & a_{77} \end{bmatrix} \begin{bmatrix} j^{+}_{xrg} \\ j^{-}_{xlg} \\ j^{+}_{yrg} \\ j^{-}_{ylg} \\ j^{+}_{zrg} \\ j^{-}_{zlg} \\ \bar{\phi}_g \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} \quad (10)$$

Note that this linear system which involves so called the arrow head matrix is quite easy to solve since it requires trivial elimination and backward substitutions.

2.2 Two-Group CMFD Formulation

The multi-group outgoing currents and the node average flux determined by the one multi-group nodal kernel can be used in the group condensation into two groups. The two-group cross sections for CMFD equation are generated by spectrum weighting as follows:

$$\varphi_g = \frac{\bar{\phi}_g}{\bar{\phi}_G}, \quad \bar{\phi}_G = \sum_{g \in G} \bar{\phi}_g \quad (G = 1,2), \quad (11)$$

$$\varsigma_{sg} = \frac{j^{out}_{sg}}{j^{out}_{sG}}, \quad j^{out}_{sG} = \sum_{g \in G} j^{out}_{sg} \quad (G = 1,2),$$

$$\Sigma_{XG} = \sum_{g \in G} \Sigma_{Xg} \varphi_g$$

In addition, the two-group corrective nodal coupling coefficients are determined by the following equation:

$$\tilde{D}_{sG} = -\frac{\hat{D}_{sG}(\bar{\phi}^R_G - \bar{\phi}^L_G) + J_{sG}}{\bar{\phi}^R_G + \bar{\phi}^L_G}, \quad (12)$$

In the CMFD problem, the two-group interface currents are represented using the corrective coefficients as the following form:

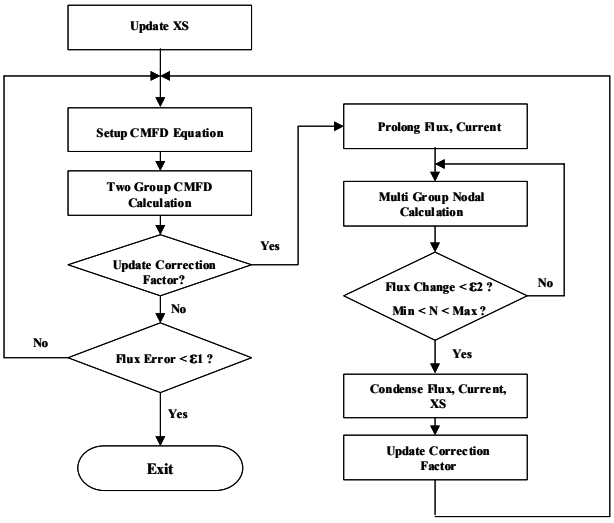
$$J_{sG} = -\hat{D}_{sG}(\bar{\phi}^R_G - \bar{\phi}^L_G) - \tilde{D}_{sG}(\bar{\phi}^R_G + \bar{\phi}^L_G) \quad (13)$$

Once a two-group CMFD solution is obtained, the multi-group flux and currents are computed from previously stored flux and outgoing current spectra, and then multi-group nodal iterations are performed.

$$\bar{\phi}_g = \varphi_g \bar{\phi}_G \ (g \in G), \quad j^{out}_{sg} = \varsigma_{sg} j^{out}_{sG} \quad (14)$$

The following flowchart represents the entire process of the alternating two-group CMFD and multi-group nodal calculations.

Figure 1: Flowchart of the nonlinear iteration scheme

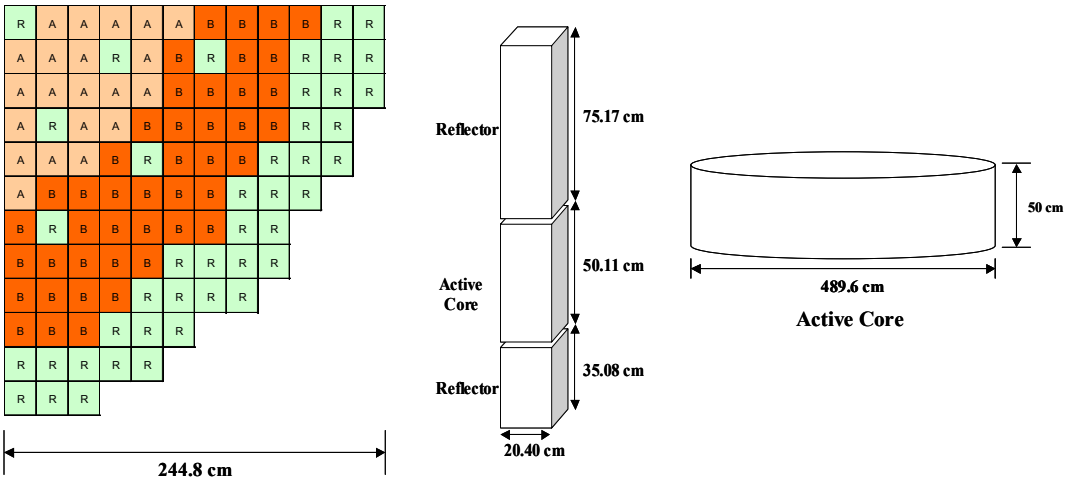


3. Numerical Results

3.1 Fast Reactor Analysis

In order to examine the efficiency of the presented multi-group nonlinear nodal method, two multi-group problems have been analyzed and the solution accuracy and the convergence characteristics were evaluated. The first model is BFS75 9-group problem modified into the rectangular geometry and the second model core is PEACER300 with 21-group, which is a fast-spectrum transmutation reactor characterized by a pan-cake type core shape.

Figure 2: PEACER reactor core and fuel assembly structure.



The eigenvalue, the number of outer iterations and computing time obtained for the first case are listed in Table 1. First, it is noted that the number of outer iterations and the number of nodal updates are reduced significantly and the calculation time as well.

Table 1: Comparison of nonlinear nodal calculation for BFS75 Problem

	Nodal Only	2-G CMFD
K_eff	1.05411	1.05411
Outer Iterations	661	18
Nodal Sweeps	661	32
Time (sec)	1053.48	74.91

Above result was based on fewer than two nodal sweeps per nodal update which was determined optimal by a previous study of the dependence of the iteration characteristics and computing time on the number of one- node sweeps per nodal update. As shown in Table 2, the computing time is reduced and the calculation efficiency is increased in the case that the number of one node sweep per nodal update is small.

In the solution of the multigroup one-node problem, the order of the group sweep and space sweep could be an important factor in the view of efficiency. Namely, one can solve the whole one node problems first for a given energy group, then move to the next group (Space Major Sweep). Or one can solve the slowing down problem for each node first then move to the other node (Energy Major Sweep). The iteration characteristics of two cases are shown in Table 3. One can find that two cases have the same iteration characteristics. But, the computational time is different, due to different program efficiency resulted from variable allocation pattern.

At last, the effect of the group split in the two-group CMFD calculation was inverstigated. The iteration characteristics for various group split patterns are given in Table 4. This result indicates that there is an optimum split pattern which is to use 8 groups for Group 1 and 13 groups for group 2, however, in fact, the dependency is weak.

Table 2: Iteration characteristics and computing time related to the number of one node sweep per nodal update

One node sweep per nodal update	Time (sec)	Iteration			K_eff
		Outer iteration	Nodal updates	Total one node sweep	
1~2	30.3	25	24	45	1.01883
1~3	30.4	19	17	47	1.01884
2~4	32.1	16	14	51	1.01884
3~5	33.8	14	12	54	1.01884
5~7	36.9	12	10	60	1.01883
7~10	39.3	12	8	65	1.01883
10~15	46.4	10	7	77	1.01882
Ref.					1.01882

Table 3: Comparison of two cases for one-node sweep

Sweep	Time (sec)	Iteration			K_eff
		Outer iteration	Nodal updates	Total one node sweep	
Energy Major	33.8	19	17	47	1.01884
Space Major	58.7	19	17	47	1.01884

Table 4: Iteration characteristics related to the number of sub-group

Group division (fast, thermal)	Time (sec)	Iteration			K_eff	Fission source Norm
		Outer iteration	Nodal updates	Total one node sweep		
(7,14)	30.0	18	14	40	1.01887	27.08567
(8,13)	27.6	17	13	36	1.01889	27.08565
(9,12)	29.1	18	14	40	1.01887	27.08572
(10,11)	31.1	18	15	43	1.01887	27.08572
(11,10)	33.1	19	16	46	1.01885	27.08581
(12, 9)	33.8	21	17	47	1.01884	27.08590
(13, 8)	33.8	22	18	52	1.01884	27.08595
(14, 7)	36.2	23	19	54	1.01882	27.08603
Ref.					1.01882	27.08601

4. Conclusion

A two-group CMFD formulation based on a multigroup one-node SANM kernel was developed. The SANM solution was obtained by solving simultaneously for the outgoing currents and the node average flux. It turned out that the two-group CMFD works quite effectively yielding a more than a factor of 14 reduction in the computing time. It was verified that the number of nodal sweeps per nodal update does not have to be large and the order of the energy sweep and the space sweep in the solution of the multigroup nodal problem is irrelevant and the group split pattern does not matter much either. The method proposed here is straightforwardly applicable to multigroup kinetics calculation. It is expected that the multigroup transient analysis which requires expensive multigroup calculations at each time step would benefit much from the improved efficiency by this method.

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