PHYSOR 2002, Seoul, Korea, October 7-10, 2002

NUMERICAL CALCULATION OF THE DELAYED-α EIGENVALUE USING A STANDARD CRITICALITY CODE

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ABSTRACT

The α-eigenvalue problem is considered taking into account delayed neutrons, as this eigenvalue represents a physically meaningful quantity for characterising a non-critical reactor as it represents the asymptotic time behaviour of the reactor considered. The delayed-α eigenvalue equation is derived and it is shown how this eigenvalue problem can be solved with limited modifications of a standard criticality code. It requires the calculation of two quantities very similar to the reactivity and the neutron generation time, respectively, and the solution of the inhour equation for the algebraically largest value. Although the delayed-α eigenvalue equation is known from the literature it is seldomly actually used nor solved for a general case.
To check the results of the general criticality code, two simple but meaningful cases are considered for which an analytic solution can be obtained, a one-group slab reactor with reflector and a bare three-group slab reactor. The results of the modified criticality code compare well with the analytical results. It is also concluded that the delayed-α eigenfunctions (spatial and energy behaviour) does not significantly differ from the *k*-eigenfunction behaviour, but differs from the prompt-α eigenfunction. Although the method it worked out for a diffusion approximation of the neutron transport, it can equally well be applied to a discrete ordinates treatment of the directional dependence in the neutron transport equation.

1. INTRODUCTION

Criticality codes calculate by default the effective multiplication factor k_{eff} of a reactor or multiplying system. This eigenvalue results as the ratio of production of neutrons over neutron losses once the flux distribution has converged to the (*k*-)eigenfunction. This eigenvalue is most frequently used to characterise a multiplying system and gives a measure of the departure from criticality. From all theoretical eigenvalue definitions it is the easiest to calculate numerically using a power method iteration technique. However, it has no strict physical meaning except for the case that $k_{\text{eff}}=1$, i.e. at criticality. Only then the calculated flux distribution will actually be realized in the reactor (as far as the system modelling is appropriate). For a non-critical system the reactor will not be stationary, the actual flux distribution will differ from the calculated *k*-eigenfunction distribution and the value of k_{eff} only qualitatively indicates the (asymptotic) time constant with which the reactor power will increase or decrease. Delayed neutrons, which play a very important role in reactor kinetics, can be taken into account in a k_{eff}-calculation using a fission spectrum appropriately averaged over prompt and delayed neutrons.
With respect to the anticipated time dependent behaviour of the neutron flux, the α-eigenvalue looks more appropriate. The α-eigenvalue equation is obtained from the time-dependent neutron transport or diffusion equation by substituting $\Phi(\boldsymbol{r},E,t)=\phi(\boldsymbol{r},E)\,\mathrm{e}^{\alpha t}$ for the neutron flux. This results in a

time-independent eigenvalue equation for the α-eigenvalue with an additional absorption term α/v with v the neutron velocity. The eigenfunction $\phi(\boldsymbol{r},E)$ differs from the k-eigenfunction. Most criticality codes allow for the calculation of the α-eigenvalue by introducing an additional absorption term α/v during the power iterations aiming at a multiplication factor k=1. The α-eigenvalue has a physical meaning as it describes the asymptotic time behaviour of the reactor power and flux distribution, provided no dynamic feedback effects occur. However, this approach does not take into account delayed neutrons, hence the physical interpretation of the α-eigenvalue is only applicable for systems that are super prompt critical. For reactors this situation should not occur because of the very fast power increase.
For a realistic physical interpretation the delayed neutrons should be taken into account, which will result in a delayed-α eigenvalue problem, still assuming an exponential time dependence.
First, the actual eigenvalue equation will be derived and it will be shown how this equation can be solved making use of a standard criticality code with a prompt-α eigenvalue option. Next, two examples will be discussed which can also be solved analytically to check the results.
Attention to the numerical solution of the delayed-α eigenvalue problem has been given by Colombo and Ravetto[1] in the framework of a study into the numerical solution of different types of eigenvalue problems using the latent roots of a matrix eigenvalue problem. However, their method is restricted to neutron transport eigenvalue problems in plane geometry with isotropic scattering. In the current paper a method is developed that can be applied to all reactor models and transport models that can be handled by an existing criticality code.

2. THEORY

In this section the theory will be treated using the energy dependent diffusion approximation for the neutron transport and application to a multigroup diffusion criticality code. However, the extensions to the direction dependent case and application to a multigroup discrete ordinates criticality code are straight forward and do not pose any principal problem.
The time and energy dependent Boltzmann equation with delayed neutrons in diffusion approximation reads[2]

$$\frac{1}{v}\frac{\partial \Phi}{\partial t} = \nabla . D(r,E)\nabla \Phi(r,E,t) - \Sigma_t(r,E)\Phi(r,E,t) + \int \Sigma_s(r,E' \to E)\Phi(r,E',t)dE' + (1-\beta)\chi_p(E)\int \nu\Sigma_f(r,E')\Phi(r,E',t)dE' + \sum_k \chi_k(E)\lambda_k C_k(r,t) \tag{1}$$

with Φ the neutron flux, D the diffusion coefficient, Σ_t the total macroscopic cross section, Σ_f the fission cross section, $\Sigma_s(E' \to E)$ the differential scattering cross section and $\chi_p(E)$ the prompt fission spectrum. The parameters of the delayed neutrons are $\beta = \sum_k \beta_k$ the total delayed neutron fraction and λ_k the decay constant of the k^{th} delayed neutron group and $\chi_k(E)$ the fission spectrum of delayed group k. C_k is the neutron precursor concentration of the k^{th} group. The equations for these concentrations are

$$\frac{\partial C_k}{\partial t} = \beta_k \int \nu\Sigma_f(r,E')\Phi(r,E',t)dE' - \lambda_k C_k(r,t) \tag{2}$$

As these equations are linear we can expect an exponential solution for the time dependence. The delayed-α eigenvalue equation is obtained when substituting the following exponential time dependence

$$\begin{aligned} \Phi(r,E,t) &= \phi(r,E)e^{\alpha t} \\ C_k(r,t) &= C_k(r)e^{\alpha t} \end{aligned} \tag{3}$$

When applying this time behaviour to Eq.(2) we can solve the delayed neutron precursor densities in terms of the neutron flux and substitute the result into Eq.(1) to obtain

$$\begin{aligned} &\nabla.D(r,E)\nabla\phi(r,E) - \left[\Sigma_t(r,E) + \frac{\alpha}{v}\right]\phi(r,E) + \int \Sigma_s(r,E' \to E)\phi(r,E')dE' \\ &+ \left\{(1-\beta)\chi_p(E) + \sum_k \frac{\beta_k \lambda_k}{\alpha+\lambda_k}\chi_k(E)\right\}\int \nu\Sigma_f(r,E')\phi(r,E')dE' = 0 \end{aligned} \tag{4}$$

Eq.(4) can be interpreted as a prompt-α eigenvalue equation with the term within bracelets a special fission spectrum, which depends on the eigenvalue α itself. Note that this term essentially differs from the formulation given in Ref. 1. Further note that the integral over energy of this fission spectrum is not normalized to unity. The difficulty of solving Eq.(4) lies in determining the value of α when iterating on the appropriate flux eigenfunction distribution. For a prompt-α iteration, i.e. with all β_k=0, a criticality code will start with a guessed value of α and obtain a new iterant depending on the value of the multiplication factor k and some interpolation of α as a function of k determined from previous iterations. For the delayed-α eigenvalue problem this dependence can be obtained in a different way. To this end we introduce the following quantities

$$\rho^* = 1 - \frac{\iint \Sigma_a \phi(r,E')dE'dV + \iint n \cdot J(r,E')dE'dS}{\iint \nu\Sigma_f(r,E')\phi(r,E')dE'dV} \tag{5}$$

$$\Lambda^* = \frac{\iint \frac{1}{v}\phi(r,E')dE'dV}{\iint \nu\Sigma_f(r,E')\phi(r,E')dE'dV} \tag{6}$$

with ϕ the flux eigenfunction as obtained from the last iteration. J in Eq.(5) is the neutron current and the surface integration is over the outer surface of the system. Therefore, the quotient in this equation represents total absorption plus leakage over the production and the right hand side is a kind of reactivity. Likewise, Eq.(6) represents a kind of neutron generation time. These quantities can be used when Eq.(4) is integrated over energy and the total reactor volume, leading to

$$\rho^* = \alpha\Lambda^* + \alpha\sum_k \frac{\beta_k}{\alpha+\lambda_k} \tag{7}$$

This equation is in fact an inhour equation[2]. For a given value of ρ^* and Λ^* the inhour equation has a number of solutions for α, from which we look for the algebraically largest value to get the fundamental mode delayed-α eigenvalue and function, which represents the asymptotic time

behaviour of the original system from Eq.(1). So, if we extend the criticality code with the calculation of the quantities ρ^* and Λ^* according to Eqs.(5) and (6) and the solution of the inhour equation for the largest value of α we automatically get the next iterate for α. For the next eigenfunction iteration the fictitious fission spectrum value is calculated from

$$\chi(E) = (1-\beta)\chi_p(E) + \sum_k \frac{\beta_k \lambda_k}{\alpha + \lambda_k} \chi_k(E) \tag{8}$$

and the absorption cross section is increased by α/v. Then the normal prompt-α iteration can be continued, until convergence is reached.
For the solution of the largest value of α from the inhour equation, we note that for $\rho^*<0$ we have $-\lambda_1<\alpha<0$ and application of the simple regula falsi leads to the required solution. If $\rho^*>0$ we have $\alpha>0$ and a Newton-Raphson iteration will do the job.

3. ANALYTICAL AND NUMERICAL TEST CASES

The traditional textbook one-group bare reactor is not of much interest here as for this reactor the *k*-, prompt-α and delayed-α eigenfunctions all have the same cosine space dependence. Therefore, we consider first a one-group reflected reactor.

3.1 SLAB REACTOR WITH REFLECTOR

The delayed-α eigenvalue equations for a one-group slab reactor with reflector becomes

$$\begin{aligned} &\text{core}: \quad D_c \nabla^2 \phi_c - (\Sigma_{ac} + \frac{\alpha}{v_c})\phi_c + (1-\beta+\sum_k \frac{\beta_k \lambda_k}{\alpha+\lambda_k})\nu\Sigma_f \phi_c = 0 \\ &\text{reflector}: \quad D_r \nabla^2 \phi_r - (\Sigma_{ar} + \frac{\alpha}{v_r})\phi_r = 0 \end{aligned} \tag{9}$$

If we assume the geometry symmetric around x=0 with the full core width equal to a and the reflector thickness equal to T, extending from $a/2$ to $b/2$ including the extrapolation distance, the solution is

$$\begin{aligned} \phi_c(x) &= \cos Bx \qquad x \le \frac{a}{2} \\ \phi_r(x) &= \frac{\cos\frac{1}{2}Ba}{\sinh \kappa T} \sinh \kappa(\frac{b}{2} - x) \quad \frac{a}{2} < x \le \frac{b}{2} \end{aligned} \tag{10}$$

where the flux is normalized to unity at the core centre. The value of the geometric buckling in the core depends on α:

$$B^2(\alpha) = (1-\beta+\sum_k \frac{\beta_k \lambda_k}{\alpha+\lambda_k})\nu\Sigma_f - \Sigma_{ac} - \frac{\alpha}{v_c} \tag{11}$$

Likewise

$$\kappa^2(\alpha) = \frac{\Sigma_{ar} + \dfrac{\alpha}{v_r}}{D_r} \tag{12}$$

Substitution of these solutions into Eq.(9) and application of continuity of current at the core-reflector interface leads to the eigenvalue equation

$$B \tan B\frac{a}{2} = \frac{D_2}{D_1}\kappa \coth \kappa T \tag{13}$$

The same equation is obtained for a *k*-eigenvalue calculation, but now both B and κ depend on α. Solving this equation for α requires a double iteration. For an initial guess of α we calculate κ from Eq.(12). Then we have to solve B iteratively from Eq.(13). For the fundamental mode solution $Ba/2<\pi/2$ is required. With the solution for B we have to find a new value for α iterating on Eq.(11). Then the total procedure can be repeated until convergence in α.
For a numerical calculation the full core width was taken as a=10 cm. The reflector width T=10 cm. The neutron cross sections are for the core Σ_a =0.82 cm^{-1} and $\nu\Sigma_f$=0.85 cm^{-1} and the diffusion coefficient D=0.5 cm. For the reflector Σ_a =0.1 cm^{-1} and D=0.8 cm. The neutron speed both for the core and the reflector was taken as v=2.2 10^5 cm/s.
Table I shows the results for the various eigenvalues calculated analytically and by the PROMETHEUS code [3], which is in fact a code system suitable for kinetics and dynamics calculations with an advanced option for depletion calculations. The heart is a 3-D diffusion code for rectangular, cylindrical and spherical geometry. The above theory for obtaining the delayed-α eigenvalue and eigenfunction was implemented in this code. The last column of the table gives the flux at the core-reflector interface, relative to the flux at the core centre. This gives an indication for the possible differences in eigenfunctions.

Table I. Numerical results for different eigenvalues for the one-group reactor

calculation	eigenvalue problem	eigenvalue	$\phi_{c\text{-}r}/\phi_{centre}$
analytical	k	1.00270	0.38365
PROMETHEUS code	k	1.00272	0.3825
analytical	prompt-α	474.3 s^{-1}	0.38100
PROMETHEUS code	prompt-α	476.3 s^{-1}	0.3811
analytical	delayed-α	0.10136 s^{-1}	0.38365
PROMETHEUS code	delayed-α	0.1027 s^{-1}	0.3825

The table shows a good agreement between the results of the analytical calculations and the numerical calculations with the PROMETHEUS code, taking into account that for a numerical code there are discretization errors and convergence errors.
A further demonstration of the physical meaning of the delayed-α eigenvalue is obtained as follows. With the kinetics option of the PROMETHEUS code the time dependence was calculated starting with a critical reactor (Σ_a =0.8223 cm^{-1}) and during 0.5 s linearly decreasing the absorption cross section to the value mentioned above. Fig. 1 shows the time behaviour for the first 20 s. After a transition period the time dependence approaches a single exponential behaviour. From the plot one can derive the asymptotic α-value as α=0.103 s^{-1}, which is in nice agreement with the delayed-α eigenvalue, considering that at t=20 s the time behaviour is not yet strictly asymptotic.

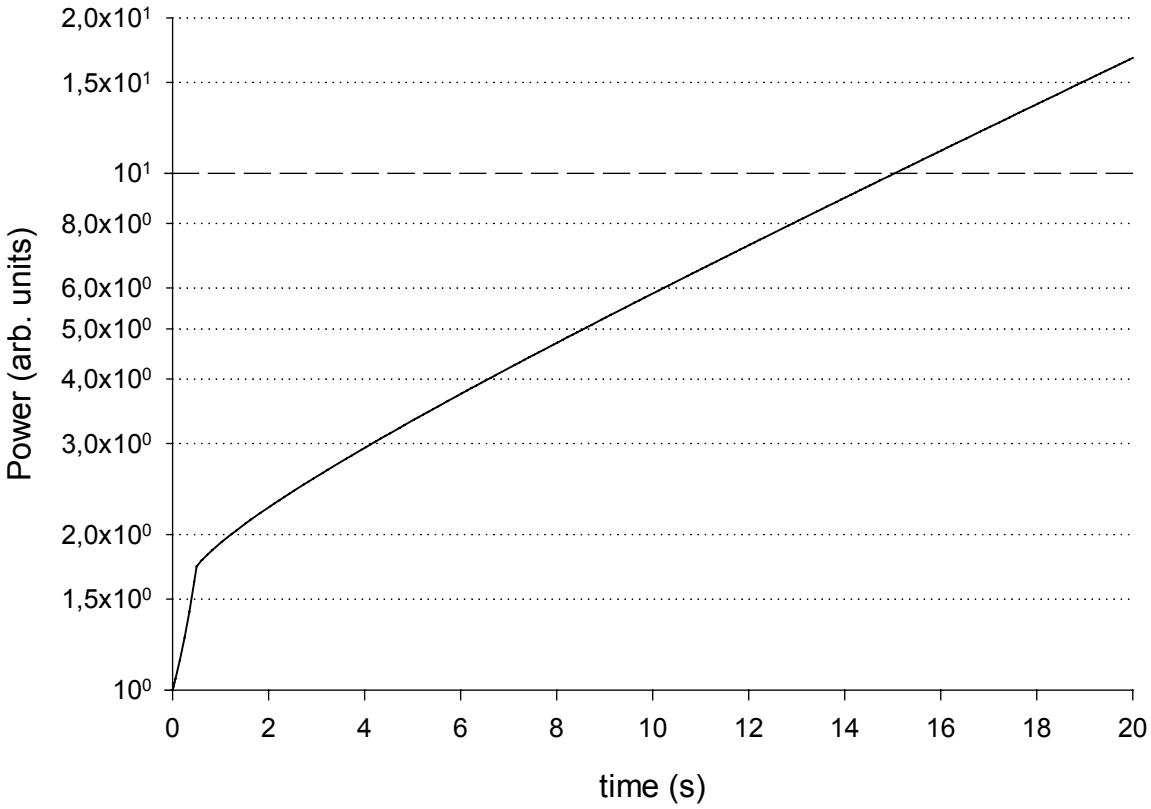

Fig. 1. Time behaviour of super-critical reactor

3.2 THREE-GROUP SLAB REACTOR

Another example that can be treated analytically and has certain interesting features is a three-group bare slab reactor. In this case we assume two fast groups with the prompt fission spectrum with fractions χ_{p1} and χ_{p2} divided over the fast groups, but with all fission neutrons released from delayed neutron precursors emitted only in the second group. This reflects the lower energy of delayed fission neutrons. For simplicity we assume further only down scattering.
For the bare slab reactor the spatial distribution of all eigenfunctions is a cosine function for all energy groups. However, the amplitudes of the group fluxes will be different for the *k*-, prompt-α and delayed-α eigenfunctions. The delayed-α eigenvalue equation for our model reads

$$
\begin{aligned}
&D_1\nabla^2\phi_1 - (\Sigma_{a1} + \frac{\alpha}{v_1} + \Sigma_{12} + \Sigma_{13})\phi_1 + (1-\beta)\chi_{p1}\sum_g \nu_g \Sigma_{fg}\phi_g = 0 \\
&D_2\nabla^2\phi_2 - (\Sigma_{a2} + \frac{\alpha}{v_2} + \Sigma_{23})\phi_2 + \Sigma_{12}\phi_1 + \left[(1-\beta)\chi_{p2} + \sum_k \frac{\beta_k\lambda_k}{\alpha+\lambda_k}\right]\sum_g \nu_g \Sigma_{fg}\phi_g = 0 \qquad (14) \\
&D_3\nabla^2\phi_3 - (\Sigma_{a3} + \frac{\alpha}{v_3})\phi_3 + \Sigma_{13}\phi_1 + \Sigma_{23}\phi_2 = 0
\end{aligned}
$$

For each group holds

$$\nabla^2\phi + B^2\phi = 0 \tag{15}$$

with $B^2=(\pi/a)^2$. Substituting this result eliminates the space dependence. Normalizing the total fission source at the centre of the reactor to unity we can simply express the flux amplitudes as a function of the eigenvalue α:

$$\begin{aligned}
\phi_1 &= \frac{1}{D_1B^2 + \Sigma_{a1} + \dfrac{\alpha}{v_1}}(1-\beta)\chi_{p1} \\
\phi_2 &= \frac{1}{D_2B^2 + \Sigma_{a2} + \dfrac{\alpha}{v_2}}\left((1-\beta)\chi_{p2} + \sum_k \frac{\beta_k\lambda_k}{\alpha+\lambda_k} + \Sigma_{12}\phi_1\right) \\
\phi_3 &= \frac{1}{D_3B^2 + \Sigma_{a3} + \dfrac{\alpha}{v_3}}\left(\Sigma_{13}\phi_1 + \Sigma_{23}\phi_2\right)
\end{aligned} \tag{16}$$

The eigenvalue α can be solved iteratively from the normalization condition

$$\nu_1\Sigma_{f1}\phi_1(\alpha) + \nu_2\Sigma_{f2}\phi_2(\alpha) + \nu_3\Sigma_{f3}\phi_3(\alpha) = 1 \tag{17}$$

For a numerical calculation we used a full slab width of a=20 cm. Cross section data are given in Table II. The set of delayed neutron data from the JEF-2 data file were used.

Table II. Neutron data for the 3-group problem

group g	D_g (cm)	Σ_a (cm^{-1})	$\Sigma_{g,g+1}$ (cm^{-1})	$\Sigma_{g,g+2}$ (cm^{-1})	$\nu_g\Sigma_{fg}$ (cm^{-1})	v_g (cm/s)	χ_{pg}
1	2.0	0.02	0.1	0.05	0.05	$1.6\ 10^9$	0.8
2	1.0	0.01	0.5	-	0.10	$8.0\ 10^8$	0.2
3	0.5	0.833	-	-	0.85	$2.2\ 10^5$	0.0

The solution of the analytical problem is shown in Table III together with the numerical solution from the PROMETHEUS code, with the neutron group fluxes renormalized to the required normalization. For the k-eigenvalue calculation the fission spectrum used was the weighed average of the prompt and delayed fission spectrum. Note that for the delayed-α eigenvalue calculation the true fractions of delayed neutrons must be used and not the effective fractions normally used in kinetics calculations.

Table III. Numerical results for different eigenvalue problems

calculation	eigenvalue problem	eigenvalue	ϕ_1	ϕ_2	ϕ_3
analytical	k	1.00321	3.6222	1.0618	0.8423
analytical	prompt-α	835.8 s^{-1}	3.6222	1.0618	0.8385
analytical	delayed-α	0.1148 s^{-1}	3.6222	1.0564	0.8391
PROMETHEUS code	delayed-α	0.1145 s^{-1}	3.622	1.0564	0.8391

As we see the PROMETHEUS criticality calculation predicts well the delayed-α eigenvalue. The differences in the eigenfunctions for the various eigenvalue problems are small.

CONCLUSIONS AND DISCUSSION

The numerical and analytical examples demonstrate that the theory of the delayed-α eigenvalue can be used to calculate the asymptotic time behaviour of a non-critical reactor. The implementation in a multigroup diffusion criticality code also demonstrates that the delayed-α eigenvalue problem can be satisfactorily solved under very general conditions of reactor configuration and neutron transport approximation.
The analytical and numerical examples also show that the prompt and delayed-α eigenvalues are totally different and that there may be differences in their respective eigenfunctions. However, the delayed-α eigenfunction is in these examples hardly distinguishable from the k-eigenfunction. The differences between these two types of eigenfunctions may be more significant for a large reactor, especially with a heavy water reflector, as for this reflector material the absorption cross section is extremely small. Nonetheless, from Eq.(14) we see that for realistic values like α=0.2 s^{-1} and v=2.2 10^5 cm/s and knowing that the thermal absorption cross section for heavy water is about 3.3 10^{-5} cm^{-1}, we cannot expect large differences, unless one treats the thermal energy range with several groups. Then for the low energy groups a noticeable difference may occur.
The numerical stability of the delayed-α eigenvalue solution method was not explicitly studied. It is important to tune the convergence criterion of the inner and outer iterations to that of the a iteration.
The one-group reflected slab reactor and especially the 3-group slab reactor can be very useful in studying k and α-eigenvalues and functions. Especially the latter problem can be used to calculate analytically parameters like the neutron generation time and the effective delayed neutron fraction as defined in space-dependent kinetic theory, using a flux and adjoint weighed quantities. Such a study, however, was not the subject of this paper.

ACKNOWLEDGEMENT

V.A. Khotylev, currently at the Canadian Nuclear Safety Commission, Ottawa, Canada, contributed to the computer code implementation of the ideas described in this paper.

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