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BOUNDARY INTEGRAL APPROACH TO NEUTRON TRANSPORT PROBLEMS

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ABSTRACT

The one-velocity, isotropic scattering neutron transport equation in general geometry is considered in a second-order form, which can be seen as the limit of the A_N approximation (or, equivalently, of the SP_{2N-1} approximation) as $N \to \infty$. Such a second-order form is rigorously equivalent to the classical integral transport equation for the particular set of problems in which the total cross section is everywhere constant (Cσ problems). The fundamental solution (Green function) of this equation is explicitly determined and is shown to involve a bilinear expansion in terms of the Case eigenfunctions. It is then possible to formulate any Cσ problem as a set of boundary integral equations and, once these equations have been solved and certain projection coefficients have been calculated, the solution of the problem can be represented, inside each homogeneous region in which the system is assumed to be subdivided, by a superposition of "elementary solutions".

1. INTRODUCTION

After forty years since the pioneering work by E.M. Gelbard [1], the simplified P_N (SP_N) method is now fully acknowledged as a practical, efficient approximate solution method for 2D and 3D transport problems. However, the error of the SP_N method does not approach zero, in general, if the order N of the approximation tends to infinity. This almost obvious consequence of the drastic simplification of the spherical harmonics expansion is an important drawback of the method and probably the cause of some past reluctance to include it among the most widely used tools of reactor physics.

Recently, it has been proved that the odd-order SP_{2N-1} approximation is equivalent to the N-th order approximation of an apparently quite different second order transport method, called A_N [2]. The proof holds for arbitrary (space dependent) total and scattering cross sections, $\sigma(\mathbf{r})$ and $\sigma_s(\mathbf{r})$, respectively; the only important restriction concerns the scattering process, which is assumed to be isotropic or linearly anisotropic [3][1]. Another result concerns a particular class of problems, for which the total cross section is everywhere constant (Cσ, "constant sigma", problems). For such problems the A$_N$-SP$_{2N-1}$ method is equivalent to the general spherical harmonics P$_{2N-1}$ method [4, 3], which implies that the A$_N$-SP$_{2N-1}$ solution approaches the exact transport solution as $N \to \infty$. The structure of the A$_N$ equations is, however, simpler than that of the original SP$_{2N-1}$ equations and allows to get a better insight into the high N behavior. For Cσ problems it is even possible to work out directly the $N \to \infty$ form of the A$_N$-SP$_{2N-1}$ theory, thus obtaining an alternative exact form of the transport equation (a very early proof can be found in ref. [5]).

The present paper gives a survey of the essential features of the A$_\infty$-SP$_\infty$ (more simply A$_\infty$) theory and illustrates some of its relations with other topics of neutron transport. Much advantage is taken from a transformation of the A$_\infty$ integro-differential equation into a boundary integral equation, since the Green kernel which appears in it contains a bilinear expansion in terms of the Case eigenfunctions. This structure is inherited by the solution, which is given therefore the form of a 3D Case expansion.

In section 2 some aspects of a central problem in particle transport, the half-space albedo problem, are revisited, in order to enucleate the ideas which originated the above developments of the A$_\infty$ theory. The derivation of the A$_\infty$ integro-differential equation is recalled in section 3, where the transformation into the boundary integral form is also performed. The explicit expression of the Green function is worked out in section 4, where it is also shown how the solution may reproduce the "elementary solution" structure of the Green function. Section 5 is devoted to a few remarks concerning the limits of the present approach.

2. CROSS-ROADS OF DIFFERENT TRANSPORT METHODS

From an abstract point of view, a motivation for the great appeal of the Case expansion method is that it allows to treat e.g. the half-space ($x > 0$) *albedo* problem as an evolution problem in the x variable. Namely, let the half space be occupied by a homogeneous scattering and absorbing medium with $\gamma = \sigma_s/\sigma < 1$ and let the m.f.p. be taken as unit length (thus, $\sigma = 1$). For $x < 0$ there is vacuum. Let us also assume that no volume sources exist in the half-space $x > 0$ and that the neutron population inside it is sustained by the angular flux $f(\mu)$, $\mu > 0$, entering from the free surface. The steady-state, isotropic scattering transport equation can be given the following form:

$$T\frac{d\phi}{dx} = -B\phi, \tag{1}$$

where $\phi = \phi(x, \mu)$ is the angular flux at x, along a direction forming an angle with the positive direction of the x-axis the cosine of which is μ, T is the multiplication operator by

[1] Unfortunately, the Publisher sent to print a preliminary version of article [3], which contained a number of misprints. A page of corrigenda will appear in a future issue. In the meantime, interested readers can obtain a list of corrections directly by mailing ravetto@polito.it.

μ, $\mu \in [-1, 1]$, and B is the collision operator

$$Bh(\mu) = h(\mu) - \frac{\gamma}{2} \int_{-1}^{1} h(\mu')d\mu'. \tag{2}$$

The albedo problem is defined by adding the conditions

$$\lim_{x \to +\infty} \phi(x, \mu) = 0 \tag{3}$$

and

$$\phi(0, \mu) = f(\mu), \qquad \mu > 0. \tag{4}$$

As the multiplication operator T is clearly invertible, Eq. (1) can be written as:

$$\frac{d\phi}{dx} = -A\phi, \tag{5}$$

where $A = T^{-1}B$, so that it is tempting to represent the solution in the semigroup form

$$\phi(x, \mu) = e^{-xA}\phi(0, \mu). \tag{6}$$

There is, however, a difficulty, namely that only the incoming angular flux, $f(\mu)$, is actually given. Obtaining the supplementary information that is needed to complete the initial datum $\phi(0, \mu)$ on the negative μ interval is quite not trivial (if the albedo problem is considered in its genuine form, the only interesting quantity is just the outgoing, i.e. reflected angular flux). It can be shown, however, that an albedo, or *extension*, operator

$$E : f(\mu) \rightarrow \phi(0, \mu) \tag{7}$$

actually exists [6] and is such that the resulting $\phi(0, \mu)$ belongs to the invariant subspace associated to the positive part of the spectrum of A, which is necessary for condition (3) to be satisfied. The solution can therefore be given in the following form:

$$\phi(x, \mu) = e^{-xA}Ef(\mu). \tag{8}$$

The explicit construction of the solution can be much facilitated if one has recourse to the partial range completeness theorem [7,8]. However, the theory underlying this theorem is not less complicate than that concerning the extension operator, being the two approaches indeed equivalent. One may then ask if the above, apparently innocent, "initial value" problem really needs such a mathematical tour-de-force. If one aims at a fully analytical, closed form treatment, the answer is decidedly affirmative. But if one is not so exacting, other solution procedures can be exploited. The one that is shown below, in particular, does not give condition (4) the role of a disguised initial value condition, but, more naturally, that of a plain boundary condition.

Some known definitions must be first recalled. Let $G\left(x', \mu' \to x, \mu\right)$ denote the infinite medium Green function for a source at x' emitting one particle per second along μ' [9,8]:

$$G\left(x', \mu' \to x, \mu\right) = \begin{cases} \dfrac{1}{N_0}\phi_{0+}(\mu)\phi_{0+}(\mu')e^{-(x-x')/\nu_0} + \displaystyle\int_0^1 \dfrac{1}{N_\nu}\phi_\nu(\mu)\phi_\nu(\mu')e^{-(x-x')/\nu}d\nu, & x' < x, \\ \dfrac{1}{N_0}\phi_{0-}(\mu)\phi_{0-}(\mu')e^{-(x'-x)/\nu_0} + \displaystyle\int_0^1 \dfrac{1}{N_\nu}\phi_{-\nu}(\mu)\phi_{-\nu}(\mu')e^{-(x'-x)/\nu}d\nu, & x' > x, \end{cases} \tag{9}$$

with

$$\phi_{0\pm}(\mu) = \frac{\gamma\nu_0}{2}\frac{1}{\nu_0 \mp \mu}, \tag{10}$$

$$\phi_\nu(\mu) = \frac{\gamma\nu}{2}P\frac{1}{\nu - \mu} + \lambda(\nu)\delta\left(\nu - \mu\right), \qquad \nu \in [-1, 1], \tag{11}$$

$$N_0 = \frac{\gamma\nu_0}{2}\left(\frac{\gamma\nu_0^2}{\nu_0^2 - 1} - 1\right), \tag{12}$$

$$N_\nu = \nu\left[\lambda^2(\nu) + \left(\frac{\gamma\pi\nu}{2}\right)^2\right], \tag{13}$$

where P indicates the principal part, $\lambda(\nu) = 1 - \gamma\nu\tanh^{-1}\nu$ and ν_0 is the positive root of

$$\frac{1}{\gamma} = \frac{\nu}{2}\ln\frac{\nu+1}{\nu-1}. \tag{14}$$

The construction of the Green function makes no reference to an initial value problem, but only to the basic idea of the "elementary solutions", as solutions which are separable with respect to the x and μ variables, other than to the related full range completeness theorem, the proof of which is simpler than for the half-range [8]. Although not allowing for a fully analytical treatment, the following well-known procedure based on Placzek's lemma [10, 8] is quite elementary. It has been widely generalized by several authors and also applied to the numerical solution of one-dimensional transport problems [11, 12, 13]. It is here recalled because it represents a good starting point for our boundary approach.

Let us still consider the half-space albedo problem. If the half-space $x < 0$ is filled up with the same material as the other half-space, the flux injected at $x = 0$ is seen by the diffusing medium, now extending over the whole space, as produced by the surface source

$$S_0(\mu) = \begin{cases} \mu f(\mu), & \mu > 0 \\ 0, & \mu < 0. \end{cases} \tag{15}$$

The angular flux generated by this source is then

$$\begin{aligned} \phi_\infty(x,\mu) &= \int_{-1}^{1} G\left(0,\mu' \to x,\mu\right) S_0(\mu')d\mu' \\ &= \int_{0}^{1} G\left(0,\mu' \to x,\mu\right) f(\mu')\mu' d\mu'. \end{aligned} \tag{16}$$

At any point $x > 0$ such a flux is obviously larger than the flux which would have been present in the case of the half-space facing the void, since some neutrons leaving the right half-space can unduly return to it, but we may "kill" such neutrons just after coming out by adding a suitable negative surface source, namely

$$S_-(\mu) = \begin{cases} -\left|\mu\right| \phi(0,\mu), & \mu < 0 \\ 0, & \mu > 0. \end{cases} \tag{17}$$

The corrected flux at any $x > 0$ is then

$$\phi(x,\mu) = \phi_\infty(x,\mu) - \int_{-1}^{0} G\left(0,\mu' \to x,\mu\right) \phi(0,\mu') \left|\mu'\right| d\mu'. \tag{18}$$

Taking the limit as $x \to 0^+$, the following integral equation for the outgoing flux is obtained:

$$\phi(0,\mu) = \phi_\infty(0,\mu) - \int_{-1}^{0} G\left(0,\mu' \to 0^+,\mu\right) \phi(0,\mu') \left|\mu'\right| d\mu', \quad \mu \in [-1,0), \tag{19}$$

where $\phi(0,\mu)$ appears instead of $\phi(0^+,\mu)$ since $\phi(x,\mu)$ is continuous for all $\mu < 0$ up to $x = 0$, where the negative source is placed; the notation 0^+ is retained for the Green function in order to single out the determination of this function that holds for $x > 0$, i.e. the first one in Eq. (9).

Owing to the fact that ν ranges on $(0,1]$ and μ, μ' on $[-1,0)$, the kernel $G\left(0,\mu' \to 0^+,\mu\right)$ of Eq. (19) has only a logarithmic singularity for $\mu = \mu' = 0$ and is therefore square summable. The solution of Eq. (19) thus exists e.g. in $L^2(-1,0)$ and, after substitution into Eq. (18), allows to obtain the solution of the half-space problem, $\phi(x,\mu)$.

We note that the role of the extension operator E is now played by Eq. (19). By recalling Eq. (16) one has, moreover,

$$\begin{aligned}\phi(x,\mu) &= \int_0^1 G\left(0,\mu' \to x,\mu\right) f(\mu')\mu' d\mu' + \int_{-1}^0 G\left(0,\mu' \to x,\mu\right) \phi(0,\mu')\mu' d\mu' \\ &= \int_{-1}^1 G\left(0,\mu' \to x,\mu\right) \phi(0,\mu')\mu' d\mu',\end{aligned} \tag{20}$$

since $\phi(0, \mu) = f(\mu)$ for $\mu > 0$. Using Eq. (9), the familiar Case expansion is found:

$$\phi(x,\mu) = a_{0+}\phi_{0+}(\mu)e^{-x/\nu_0} + \int_0^1 A_\nu \phi_\nu(\mu) e^{-x/\nu} d\nu, \tag{21}$$

where

$$\begin{aligned}a_{0+} &= \frac{1}{N_0}\int_{-1}^1 \phi(0,\mu')\phi_{0+}(\mu')\mu' d\mu', \\ A_\nu &= \frac{1}{N_\nu}\int_{-1}^1 \phi(0,\mu')\phi_\nu(\mu')\mu' d\mu'.\end{aligned} \tag{22}$$

Equation (8) has been recovered in the spectral form (21) without even mentioning the semigroup. Moreover, the above procedure can be easily generalized to more dimensions, at least in principle [11-13]. There is, however, a considerable difficulty. The 3D analogues of Eqs. (18) and (19) involve the 3D transport Green function $G(\mathbf{r}', \hat{\Omega}' \to \mathbf{r}, \hat{\Omega})$, where $\mathbf{r}$, $\mathbf{r}'$ are the space variables (the flux point and the source point, respectively) and the unit vectors $\hat{\Omega}$, $\hat{\Omega}'$ replace μ, μ' in specifying the direction of the particles. Unfortunately, even if an explicit representation of this function can be given, the singular character of the uncollided and first collided neutron flux [14] is an obstacle to the full exploitation of the boundary source method in its most general form. On the contrary, for Cσ problems, the second order A$_\infty$ approach allows to obtain a fairly simple Green function, as it is shown in section 4. In this section a somewhat different problem, namely a half-space problem with a distribution of isotropic volume sources $S(x)$, is also considered. In this case Eq. (18) still holds, with the only difference that the free term, Eq. (16), is replaced by the following one:

$$\phi_\infty(x,\mu) = \frac{1}{2}\int_0^\infty \int_{-1}^1 G\left(x',\mu' \to x,\mu\right) S(x')dx'd\mu'. \tag{23}$$

3. THE A$_\infty$ INTEGRODIFFERENTIAL EQUATION AND ITS BOUNDARY INTEGRAL FORM

Let the body U either be extending over the whole space or be finite, convex and surrounded by vacuum. In either case, let the total cross section σ be everywhere constant on U, which is therefore a "Cσ body". Note that the second alternative can be reduced to a particular case of the first one by simply replacing the vacuum by a purely absorbing medium with the same total cross section as the body it surrounds.

If the scattering is isotropic, the scalar flux $\Phi(\mathbf{r})$ in U is given by the Peierls integral equation

$$\Phi(\mathbf{r}) = \int_U \frac{e^{-|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|^2}\,[\gamma(\mathbf{r}')\Phi(\mathbf{r}')+S(\mathbf{r}')]\,dV', \tag{24}$$

where $\gamma(\mathbf{r}')=\sigma_s(\mathbf{r})/\sigma$ is the mean number of secondaries per collision, assumed to be less than unity for any $\mathbf{r}$, $S(\mathbf{r})$ the isotropic source density and the total m.f.p. is taken as unit length, as in the previous section. The formula

$$\frac{e^{-r}}{4\pi r^2} = \frac{1}{4\pi r}\int_0^1 \frac{e^{-r/\mu}}{\mu^2}d\mu \simeq \frac{1}{4\pi r}\sum_{\alpha=1}^{N}\frac{w_\alpha}{\mu_\alpha^2}e^{-r/\mu_\alpha}, \tag{25}$$

where μ_α and w_α denote the points and weights of a quadrature formula, is the starting point of the approximate A_N method [2,3]. The $N=\infty$ (or A_∞) formulation of this method relies on the first part of Eq. (25), where the dummy variable is given a continuum of values between 0 and 1. This suggests to represent the transport kernel of Eq. (24) as a linear superposition of diffusion kernels:

$$\Phi(\mathbf{r}) = \int_0^1 d\mu \int_U \psi^*(|\mathbf{r}-\mathbf{r}'|\,,\mu)\,[\gamma(\mathbf{r}')\Phi(\mathbf{r}')+S(\mathbf{r}')]\,dV', \tag{26}$$

where

$$\psi^*(|\mathbf{r}-\mathbf{r}'|\,,\mu) = \frac{e^{-|\mathbf{r}-\mathbf{r}'|/\mu}}{4\pi\mu^2\,|\mathbf{r}-\mathbf{r}'|} \tag{27}$$

is the solution, vanishing at infinity, of the following equation:

$$\mu^2\nabla^2\psi^* - \psi^* + \delta\,(\mathbf{r}-\mathbf{r}') = 0. \tag{28}$$

It is convenient to set

$$F(\mathbf{r},\mu) = \int_U \psi^*(|\mathbf{r}-\mathbf{r}'|\,,\mu)\,[\gamma(\mathbf{r}')\Phi(\mathbf{r}')+S(\mathbf{r}')]\,dV', \tag{29}$$

so that, according to Eq. (26),

$$\Phi(\mathbf{r}) = \int_0^1 F(\mathbf{r},\mu)d\mu. \tag{30}$$

By applying the operator $\mu^2\nabla^2$ to both sides of Eq. (29) and using Eqs. (28) and (30), the

A_∞ integrodifferential equation is obtained [5]:

$$\mu^2\nabla^2 F(\mathbf{r},\mu) - F(\mathbf{r},\mu) + \gamma(\mathbf{r})\int_0^1 F(\mathbf{r},\mu')d\mu' + S(\mathbf{r}) = 0. \tag{31}$$

This equation is equivalent to Eq. (24) and is therefore as rigorous as the latter equation, within the restrictive assumptions made above.

As concerns Eq. (31), the essential point is that the usual directional meaning of μ is completely lost, at least for 2D and 3D problems (in the 1D case such a meaning is recovered, since F can be shown to coincide with the even-parity flux apart from a factor 2 [5]). Equation (31) could be rather considered, at least formally, as a kind of energy-dependent diffusion equation, where μ stands for the neutron energy, $D(\mu) = \mu^2$ is the diffusion coefficient, the removal cross section is equal to unity and the integral term represents the (continuous) energy transfer by collisions. The quantity $F(\mathbf{r},\mu)$ will therefore be called "pseudo-flux" and its gradient, multiplied by μ^2, "pseudo-(vector) current". However, the name "pseudo-current" will be also given to $\mu^2\partial F/\partial n$, actually the normal component of this vector at a point of a surface.

The standard procedure of the boundary element method is now applied to Eq. (31). A restrictive assumption should be introduced from the outset, however, namely that the body U where the transport process occurs is made of a finite number of homogeneous regions, so that for the i-th region the scattering cross section, and consequently γ, has a constant value σ_{si} (respectively, γ_i). Of course, the assumption that the total cross section is everywhere constant must always hold, so that all the regions must have the same value of σ (with $\sigma = 1$).

Let V be a region of the system and let Σ be its boundary, assumed to be piecewise smooth. A necessary step of the procedure is to obtain an explicit expression of the fundamental solution (or Green function) to be associated to it, i.e. the solution, assumed to be vanishing at infinity, of the following equation

$$\mu^2\nabla^2\psi(\mathbf{r},\mu) - \psi(\mathbf{r},\mu) + \gamma\int_0^1 \psi(\mathbf{r},\eta)d\eta + \delta(\mathbf{r}-\mathbf{r}')\delta(\mu-\mu') = 0, \tag{32}$$

where γ refers to the material of V. ψ is clearly a radial function, i.e. it depends only on $|\mathbf{r}-\mathbf{r}'|$, the distance of the point $\mathbf{r}$ from the source point $\mathbf{r}'$. As customary, to point out the dependence on the source parameters, the Green function will be denoted henceforth by the more detailed expression $\psi(|\mathbf{r}-\mathbf{r}'|,\mu,\mu')$. The explicit form of this function is derived in the next section. Here we make use of this fundamental solution, whose existence is momentarily taken for granted, in order to transform the A_∞ integrodifferential equation into a boundary integral equation.

Let

$$c(\mathbf{r}) = \int_V \delta(\mathbf{r}-\mathbf{r}')dV' \tag{33}$$

be the characteristic function of V, equal to 0, 1, $1/2$ according to whether $\mathbf{r}$ is outside V, inside V or is a smooth point, say $\mathbf{r}_\Sigma$, of the boundary Σ, respectively. If $\mathbf{r}_\Sigma$ is non-smooth (a vertex, or an edge point), then $c = \Theta/(4\pi)$, Θ being the solid angle subtended by the tangent cone at $\mathbf{r}_\Sigma$.

Equation (31) is then written with $\mathbf{r}$ replaced by $\mathbf{r}'$, and after being multiplied by the fundamental solution $\psi(|\mathbf{r}-\mathbf{r}'|, \mu, \mu')$, is integrated with respect to $\mathbf{r}'$ over V. By applying the Green identity and using Eqs. (32) and (33), one gets, after a little algebra,

$$\begin{aligned} &-\gamma \int_V F(\mathbf{r}', \mu) \left[\int_0^1 \psi(|\mathbf{r}-\mathbf{r}'|, \eta, \mu') d\eta \right] dV' - c(\mathbf{r}) F(\mathbf{r}, \mu) \, \delta(\mu - \mu') \\ &+ \int_\Sigma \left[\psi(|\mathbf{r}-\mathbf{r}'_\Sigma|, \mu, \mu') \mu^2 \frac{\partial F(\mathbf{r}'_\Sigma, \mu)}{\partial n'} - \mu^2 \frac{\partial \psi(|\mathbf{r}-\mathbf{r}'_\Sigma|, \mu, \mu')}{\partial n'} F(\mathbf{r}'_\Sigma, \mu) \right] d\Sigma' \\ &+\gamma \int_V \psi(|\mathbf{r}-\mathbf{r}'|, \mu, \mu') \left[\int_0^1 F(\mathbf{r}', \eta) \, d\eta \right] dV' + \int_V \psi(|\mathbf{r}-\mathbf{r}'|, \mu, \mu') S(\mathbf{r}') dV' = 0, \end{aligned} \tag{34}$$

where $\mathbf{n}'$ is the outward normal at the boundary point $\mathbf{r}'_\Sigma$.

A further integration with respect to μ on $(0, 1]$, followed by an interchange of the variables μ and μ', paying due attention to the fact that ψ is symmetric with respect to μ and μ' (see later on, Eq. (50)), leads to the following, basic relationship:

$$\begin{aligned} c(\mathbf{r}) F(\mathbf{r}, \mu) - \int_0^1 d\mu' \int_\Sigma &\left[\psi(|\mathbf{r}-\mathbf{r}'_\Sigma|, \mu, \mu') \mu'^2 \frac{\partial F(\mathbf{r}'_\Sigma, \mu')}{\partial n'} \right. \\ &\left. - \mu'^2 \frac{\partial \psi(|\mathbf{r}-\mathbf{r}'_\Sigma|, \mu, \mu')}{\partial n'} F(\mathbf{r}'_\Sigma, \mu') \right] d\Sigma' = Q(\mathbf{r}, \mu), \end{aligned} \tag{35}$$

where

$$Q(\mathbf{r}, \mu) = \int_V \left[\int_0^1 \psi(|\mathbf{r}-\mathbf{r}'|, \mu, \mu') d\mu' \right] S(\mathbf{r}') dV'. \tag{36}$$

The importance of the above equation stems from the fact that, if the boundary values of the pseudo-flux, $F(\mathbf{r}_\Sigma, \mu)$, and of the pseudo-current, $\partial F(\mathbf{r}_\Sigma, \mu)/\partial n$, are known, it is possible to obtain $F(\mathbf{r}, \mu)$ at any interior point. Let now $\mathbf{r}$ be taken on the boundary, so that $\mathbf{r} \equiv \mathbf{r}_\Sigma$. Then Eq. (35) reads:

$$\begin{aligned} c(\mathbf{r}_\Sigma) F(\mathbf{r}_\Sigma, \mu) - \int_0^1 d\mu' \int_\Sigma &\left[\psi(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|, \mu, \mu') \mu'^2 \frac{\partial F(\mathbf{r}'_\Sigma, \mu')}{\partial n'} \right. \\ &\left. - \mu'^2 \frac{\partial \psi(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|, \mu, \mu')}{\partial n'} F(\mathbf{r}'_\Sigma, \mu') \right] d\Sigma' = Q(\mathbf{r}_\Sigma, \mu). \end{aligned} \tag{37}$$

If we suppose that either the boundary pseudo-flux $F(\mathbf{r}_\Sigma, \mu)$, or the boundary pseudo-current, $\mu^2 \partial F(\mathbf{r}_\Sigma, \mu)/\partial n$, is assigned (which amounts to say that either a Dirichlet or a Neumann condition, respectively, is imposed at the boundary), then Eq. (37) is an integral equation for the remaining quantity, $\mu^2 \partial F(\mathbf{r}_\Sigma, \mu)/\partial n$ or $F(\mathbf{r}_\Sigma, \mu)$, to be determined. Once the pseudo-flux and the pseudo-current are known on the boundary, one goes back to Eq. (35) and the problem is completely solved.

Although involving a second-order formulation, the present approach shows a striking resemblance with the boundary source approach of section 2: a boundary integral equation is first solved, ignoring the interior points, then the flux values at such points are readily obtained by a simple quadrature.

4. THE FUNDAMENTAL SOLUTION AND THE 3D EIGENFUNCTION EXPANSION

Equation (32) is now solved in the 1D, plane parallel version:

$$\mu^2 \frac{\partial^2 \psi_{pl}(x,\mu)}{\partial x^2} - \psi_{pl}(x,\mu) + \gamma \int_0^1 \psi_{pl}(x,\eta) d\eta + \delta(x)\, \delta(\mu-\mu') = 0, \qquad x \in \Re^1, \quad \mu, \mu' \in (0,1], \tag{38}$$

where the plane source is placed at $x = 0$ and the dependence on the parameter μ' is again understood. The above second-order equation could be identified with the even parity form of the plane geometry transport equation, except for the source term which does not resemble a symmetric function of μ. However, both μ and μ' in Eq. (38) range on the interval $(0, 1]$, so that we may replace the above source term with $\delta(x)\,[\delta(\mu-\mu') + \delta(\mu+\mu')]$ without introducing any real change, since $\delta(\mu+\mu')$ is identically zero. If the variable μ is allowed to range on the $[-1, 1]$ interval, it is readily seen that Eq. (38), with the new source (which is symmetrical with respect to μ on $[-1, 1]$) is now completely identical to the well-known second-order equation that is obtained from the first-order transport equation:

$$\mu \frac{\partial \phi(x,\mu)}{\partial x} + \phi(x,\mu) = \frac{\gamma}{2} \int_{-1}^1 \phi(x,\eta) d\eta + \delta(x)\,[\delta(\mu-\mu') + \delta(\mu+\mu')], \qquad x \in \Re^1, \quad \mu \in [-1,1], \quad \mu' \in (0,1], \tag{39}$$

by taking the even and odd parts of $\phi(x, \mu)$ [15, 16, 17].

If the even-parity component of the angular flux,

$$\phi^+(x,\mu) = \frac{1}{2}\left[\phi(x,\mu) + \phi(x,-\mu)\right], \tag{40}$$

is considered, it is also immediately recognized that ψ_{pl} coincides with the restriction of ϕ^+ to the half interval $(0, 1]$. On the other hand, the solution $\phi(x, \mu)$ of Eq. (39) is the sum of the following Green functions:

$$\phi(x,\mu) = G(0,\mu' \to x,\mu) + G(0,-\mu' \to x,\mu), \qquad \mu \in [-1,1]. \tag{41}$$

Equation (40) then implies that

$$\phi^+(x,\mu) = \frac{1}{2}\left[G(0,\mu' \to x,\mu) + G(0,-\mu' \to x,\mu) + G(0,\mu' \to x,-\mu) + G(0,-\mu' \to x,-\mu)\right]. \tag{42}$$

Substituting expressions (9) for $G\left(0, \pm\mu' \to x, \pm\mu\right)$ and observing that $\phi_{0+}\left(\mu\right) = \phi_{0-}\left(-\mu\right)$ and $\phi_{\nu}\left(\mu\right) = \phi_{-\nu}\left(-\mu\right)$, after a few manipulations we obtain:

$$\begin{aligned}\phi^{+}\left(x,\mu\right) &= \frac{1}{2N_0}\left[\phi_{0+}\left(\mu\right)+\phi_{0-}\left(\mu\right)\right]\left[\phi_{0+}\left(\mu'\right)+\phi_{0-}\left(\mu'\right)\right]e^{-|x|/\nu_0} \\ &+\int_0^1 \frac{1}{2N_\nu}\left[\phi_{\nu}\left(\mu\right)+\phi_{-\nu}\left(\mu\right)\right]\left[\phi_{\nu}\left(\mu'\right)+\phi_{-\nu}\left(\mu'\right)\right]e^{-|x|/\nu}d\nu,\end{aligned} \tag{43}$$

where the distinction between the cases $x > 0$ and $x < 0$ is no longer necessary. As ψ_{pl} is the restriction of ϕ^{+} to the positive μ's, by introducing the following functions:

$$\psi_0(\mu) = \frac{\mu}{\sqrt{\nu_0 N_0}}\left[\phi_{0+}(\mu)+\phi_{0-}(\mu)\right] = \frac{\mu}{\sqrt{\nu_0 N_0}}\left[\phi_{0+}(\mu)+\phi_{0+}(-\mu)\right], \tag{44}$$

$$\psi_\nu(\mu) = \frac{\mu}{\sqrt{\nu N_\nu}}\left[\phi_{\nu}(\mu)+\phi_{-\nu}(\mu)\right] = \frac{\mu}{\sqrt{\nu N_\nu}}\left[\phi_{\nu}(\mu)+\phi_{\nu}(-\mu)\right], \tag{45}$$

we get the following compact expression for ψ_{pl}:

$$\psi_{pl}(\left|x-x'\right|,\mu,\mu') = \frac{\nu_0}{2}\frac{\psi_0(\mu)\psi_0(\mu')}{\mu\mu'}e^{-|x-x'|/\nu_0} + \int_0^1 \frac{\nu}{2}\frac{\psi_\nu(\mu)\psi_\nu(\mu')}{\mu\mu'}e^{-|x-x'|/\nu}d\nu, \quad x,x' \in \Re^1;\ \mu,\mu' \in (0,1], \tag{46}$$

where, for the sake of generality, the δ source is now placed at an arbitrary abscissa x'. For $\gamma = 0$ the solution of Eq. (38) takes a much simpler form, namely:

$$\psi_{pl} = \psi^*_{pl}(\left|x-x'\right|,\mu)\delta\left(\mu-\mu'\right), \tag{47}$$

where $\psi^*_{pl}(\left|x-x'\right|,\mu)$ is the simple diffusion kernel

$$\psi^*_{pl}(\left|x-x'\right|,\mu) = \frac{1}{2\mu}e^{-|x-x'|/\mu}. \tag{48}$$

The Green function for a point source at $\mathbf{r}'$ can be obtained by means of the well-known relationship between general point and plane Green functions

$$\psi(u,\mu,\mu') = -\frac{1}{2\pi u}\frac{\partial\psi_{pl}(u,\mu,\mu')}{\partial u}, \tag{49}$$

where u is to be interpreted as the distance of the points $\mathbf{r}$, $\mathbf{r}'$, resp. x, x' [8]. Thus:

$$\psi(|\mathbf{r}-\mathbf{r}'|,\mu,\mu') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\left[\frac{\psi_0(\mu)\psi_0(\mu')}{\mu\mu'}e^{-|\mathbf{r}-\mathbf{r}'|/\nu_0} + \int_0^1 \frac{\psi_\nu(\mu)\psi_\nu(\mu')}{\mu\mu'}e^{-|\mathbf{r}-\mathbf{r}'|/\nu}d\nu\right],$$
$$\mathbf{r},\mathbf{r}' \in \Re^3; \qquad \mu,\mu' \in (0,1]\,. \tag{50}$$

By exploiting the orthogonality relations of the Case eigenfunctions, the following orthonormality relations for $\psi_0(\mu)$ and $\psi_\nu(\mu)$, $\nu \in (0,1]$, are easily obtained:

$$\int_0^1 \psi_0^2(\mu)d\mu = 1, \tag{51}$$

$$\int_0^1 \psi_0(\mu)\psi_\nu(\mu)d\mu = 0, \tag{52}$$

$$\int_0^1 \psi_{\nu'}(\mu)\psi_\nu(\mu)d\mu = \delta(\nu-\nu')\,. \tag{53}$$

The following results are also useful

$$\int_0^1 \psi_0(\mu)\frac{d\mu}{\mu} = \frac{1}{\sqrt{\nu_0 N_0}}, \qquad \int_0^1 \psi_\nu(\mu)\frac{d\mu}{\mu} = \frac{1}{\sqrt{\nu N_\nu}}. \tag{54}$$

If $\gamma = 0$, the solution of Eq. (32) again undergoes a considerable simplification:

$$\psi(|\mathbf{r}-\mathbf{r}'|,\mu,\mu') = \psi^*(|\mathbf{r}-\mathbf{r}'|,\mu)\delta(\mu-\mu')\,, \tag{55}$$

where ψ^* is the diffusion kernel of Eq. (27).

Let us abbreviate our notations by setting:

$$\Psi_\mu(|\mathbf{r}-\mathbf{r}'|) = \frac{e^{-|\mathbf{r}-\mathbf{r}'|/\mu}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{56}$$

and replace $\psi(|\mathbf{r}-\mathbf{r}'|,\mu,\mu')$ by the following kernel:

$$\begin{aligned}\Psi(|\mathbf{r}-\mathbf{r}'|,\mu,\mu') &= \mu\mu'\psi(|\mathbf{r}-\mathbf{r}'|,\mu,\mu')\\ &= \psi_0(\mu)\psi_0(\mu')\Psi_0(|\mathbf{r}-\mathbf{r}'|) + \int_0^1 \psi_\nu(\mu)\psi_\nu(\mu')\Psi_\nu(|\mathbf{r}-\mathbf{r}'|)d\nu,\end{aligned} \tag{57}$$

where Ψ_0 is an abbreviation for Ψ_{ν_0}.

We now return to the boundary integral equations of section 3, Eqs. (35) and (37).

Multiplying Eq. (35) by μ and setting

$$\xi(\mathbf{r},\mu) = \mu F(\mathbf{r},\mu), \tag{58}$$

$$\tau(\mathbf{r},\mu) = \mu Q(\mathbf{r},\mu), \tag{59}$$

this equation takes the following, more symmetric form:

$$\begin{aligned} c(\mathbf{r})\xi(\mathbf{r},\mu) - \int_0^1 d\mu' \int_\Sigma d\Sigma' \Bigg[\Psi(\left|\mathbf{r}-\mathbf{r}'_\Sigma\right|,\mu,\mu') \frac{\partial \xi\left(\mathbf{r}'_\Sigma,\mu'\right)}{\partial n'} \\ - \frac{\partial \Psi(\left|\mathbf{r}-\mathbf{r}'_\Sigma\right|,\mu,\mu')}{\partial n'} \xi\left(\mathbf{r}'_\Sigma,\mu'\right) \Bigg] = \tau(\mathbf{r},\mu). \end{aligned} \tag{60}$$

As it concerns the new source term, the introduction of Eq. (57) into Eq. (36) gives, by the aid of Eqs. (54) and (59),

$$\tau(\mathbf{r},\mu) = \psi_0(\mu) Q_0(\mathbf{r}) + \int_0^1 \psi_\nu(\mu) Q_\nu(\mathbf{r}) d\nu, \tag{61}$$

where

$$Q_0(\mathbf{r}) = \frac{1}{\sqrt{\nu_0 N_0}} \int_V \Psi_0(|\mathbf{r}-\mathbf{r}'|) S(\mathbf{r}') dV', \tag{62}$$

$$Q_\nu(\mathbf{r}) = \frac{1}{\sqrt{\nu N_\nu}} \int_V \Psi_\nu(|\mathbf{r}-\mathbf{r}'|) S(\mathbf{r}') dV'. \tag{63}$$

Let expression (57) be introduced also into Eq. (60), where $\mathbf{r}$ is an interior point, so that $c(\mathbf{r}) = 1$. One obtains:

$$\begin{aligned} \xi(\mathbf{r},\mu) = \psi_0(\mu) \int_\Sigma \left[\Psi_0(|\mathbf{r}-\mathbf{r}'_\Sigma|) \frac{\partial A_0\left(\mathbf{r}'_\Sigma\right)}{\partial n'} - \frac{\partial \Psi_0(|\mathbf{r}-\mathbf{r}'_\Sigma|)}{\partial n'} A_0\left(\mathbf{r}'_\Sigma\right) \right] d\Sigma' \\ + \int_0^1 d\nu \psi_\nu(\mu) \int_\Sigma \left[\Psi_\nu(|\mathbf{r}-\mathbf{r}'_\Sigma|) \frac{\partial A_\nu\left(\mathbf{r}'_\Sigma\right)}{\partial n'} - \frac{\partial \Psi_\nu(|\mathbf{r}-\mathbf{r}'_\Sigma|)}{\partial n'} A_\nu\left(\mathbf{r}'_\Sigma\right) \right] d\Sigma' \\ + \psi_0(\mu) Q_0(\mathbf{r}) + \int_0^1 d\nu \psi_\nu(\mu) Q_\nu(\mathbf{r}), \end{aligned} \tag{64}$$

where, in general,

$$A_0\left(\mathbf{r}\right) = \int_0^1 \psi_0(\mu) \xi(\mathbf{r},\mu) d\mu, \tag{65}$$

$$A_\nu(\mathbf{r}) = \int_0^1 \psi_\nu(\mu)\xi(\mathbf{r},\mu)d\mu, \qquad \nu \in (0,1], \tag{66}$$

and also, letting $\mathbf{r}$ approach a boundary point $\mathbf{r}_\Sigma$,

$$\frac{\partial A_0(\mathbf{r}_\Sigma)}{\partial n} = \int_0^1 \psi_0(\mu)\frac{\partial \xi(\mathbf{r}_\Sigma,\mu)}{\partial n}d\mu, \tag{67}$$

$$\frac{\partial A_\nu(\mathbf{r}_\Sigma)}{\partial n} = \int_0^1 \psi_\nu(\mu)\frac{\partial \xi(\mathbf{r}_\Sigma,\mu)}{\partial n}d\mu. \tag{68}$$

If we introduce the following compact notation for the Σ integrals in Eq. (64):

$$B_0(\mathbf{r}) = \int_\Sigma \left[\Psi_0(|\mathbf{r}-\mathbf{r}'_\Sigma|)\frac{\partial A_0(\mathbf{r}'_\Sigma)}{\partial n'} - \frac{\partial \Psi_0(|\mathbf{r}-\mathbf{r}'_\Sigma|)}{\partial n'}A_0(\mathbf{r}'_\Sigma)\right]d\Sigma', \tag{69}$$

$$B_\nu(\mathbf{r}) = \int_\Sigma \left[\Psi_\nu(|\mathbf{r}-\mathbf{r}'_\Sigma|)\frac{\partial A_\nu(\mathbf{r}'_\Sigma)}{\partial n'} - \frac{\partial \Psi_\nu(|\mathbf{r}-\mathbf{r}'_\Sigma|)}{\partial n'}A_\nu(\mathbf{r}'_\Sigma)\right]d\Sigma', \tag{70}$$

the above equation becomes:

$$\xi(\mathbf{r},\mu) = [B_0(\mathbf{r}) + Q_0(\mathbf{r})]\,\psi_0(\mu) + \int_0^1 [B_\nu(\mathbf{r}) + Q_\nu(\mathbf{r})]\,\psi_\nu(\mu)d\nu, \tag{71}$$

which shows that $\xi(\mathbf{r},\mu)$ is a linear combination of the Case-like eigenfunctions ψ_0 and ψ_ν. The coefficients of such expansion are completely determined if the boundary data $\xi(\mathbf{r}_\Sigma,\mu)$, $\partial\xi(\mathbf{r}_\Sigma,\mu)/\partial n$ and the volume source $S(\mathbf{r})$ are given. But it can be shown that the boundary data can be restricted to those which are strictly necessary, i.e. to either $\xi(\mathbf{r}_\Sigma,\mu)$ (Dirichlet boundary condition) or $\partial\xi(\mathbf{r}_\Sigma,\mu)/\partial n$ (Neumann boundary condition). Whichever set of data is chosen, the remaining set can in fact be obtained by letting $\mathbf{r}$ in Eq. (60) approach once more a boundary point $\mathbf{r}_\Sigma$, which transforms the latter equation into a boundary integral equation:

$$\begin{aligned} c(\mathbf{r}_\Sigma)\,\xi(\mathbf{r}_\Sigma,\mu) - \int_0^1 d\mu' \int_\Sigma \Bigg[\Psi(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|,\mu,\mu')\frac{\partial \xi(\mathbf{r}'_\Sigma,\mu')}{\partial n'} \\ - \frac{\partial \Psi(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|,\mu,\mu')}{\partial n'}\xi(\mathbf{r}'_\Sigma,\mu')\Bigg]d\Sigma' = \tau(\mathbf{r}_\Sigma,\mu). \end{aligned} \tag{72}$$

Although this equation could be solved as it stands, it is interesting to take the scalar product of Eq. (71) with ψ_0 and ψ_ν, and use the orthonormality relations (51-53). One gets simply:

$$A_0(\mathbf{r}) = B_0(\mathbf{r}) + Q_0(\mathbf{r}), \tag{73}$$

$$A_\nu(\mathbf{r}) = B_\nu(\mathbf{r}) + Q_\nu(\mathbf{r}). \tag{74}$$

If we recall Eqs. (69, 70), we see that the above equations can be interpreted as integral relations, similar to Eq. (60), holding separately for the A_0 and A_ν components of ξ:

$$A_0(\mathbf{r}) - \int\limits_\Sigma \left[\Psi_0(|\mathbf{r}-\mathbf{r}'_\Sigma|)\frac{\partial A_0\left(\mathbf{r}'_\Sigma\right)}{\partial n'} - \frac{\partial \Psi_0(|\mathbf{r}-\mathbf{r}'_\Sigma|)}{\partial n'}A_0\left(\mathbf{r}'_\Sigma\right)\right] d\Sigma' = Q_0(\mathbf{r}), \tag{75}$$

$$A_\nu\left(\mathbf{r}\right) - \int\limits_\Sigma \left[\Psi_\nu(|\mathbf{r}-\mathbf{r}'_\Sigma|)\frac{\partial A_\nu\left(\mathbf{r}'_\Sigma\right)}{\partial n'} - \frac{\partial \Psi_\nu(|\mathbf{r}-\mathbf{r}'_\Sigma|)}{\partial n'}A_\nu\left(\mathbf{r}'_\Sigma\right)\right] d\Sigma' = Q_\nu(\mathbf{r}). \tag{76}$$

Let the Neumann condition be chosen, so that $\partial A_0\left(\mathbf{r}_\Sigma\right)/\partial n$ and $\partial A_\nu\left(\mathbf{r}_\Sigma\right)/\partial n$ are known, via Eqs. (67, 68). Then one sets $\mathbf{r} = \mathbf{r}_\Sigma$, which transforms the above equations into the following boundary integral equations for A_0 and A_ν:

$$c(\mathbf{r}_\Sigma)A_0\left(\mathbf{r}_\Sigma\right) - \int\limits_\Sigma \left[\Psi_0(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|)\frac{\partial A_0\left(\mathbf{r}'_\Sigma\right)}{\partial n'} - \frac{\partial \Psi_0(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|)}{\partial n'}A_0\left(\mathbf{r}'_\Sigma\right)\right] d\Sigma' = Q_0(\mathbf{r}_\Sigma), \tag{77}$$

$$c(\mathbf{r}_\Sigma)A_\nu\left(\mathbf{r}_\Sigma\right) - \int\limits_\Sigma \left[\Psi_\nu(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|)\frac{\partial A_\nu\left(\mathbf{r}'_\Sigma\right)}{\partial n'} - \frac{\partial \Psi_\nu(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|)}{\partial n'}A_\nu\left(\mathbf{r}'_\Sigma\right)\right] d\Sigma' = Q_\nu(\mathbf{r}_\Sigma), \tag{78}$$

with $\nu \in (0,1]$ (the c factor has been restored). In the case of the Dirichlet condition, Eqs. (65, 66) directly yield A_0 and A_ν. Once these quantities have been obtained, Eq. (71) gives, taking account of Eqs. (73, 74),

$$\xi(\mathbf{r},\mu) = A_0(\mathbf{r})\psi_0(\mu) + \int_0^1 A_\nu\left(\mathbf{r}\right)\psi_\nu(\mu)d\nu. \tag{79}$$

Thus, by Eq. (58), we get the following Case-like expansion of $F(\mathbf{r},\mu)$:

$$F(\mathbf{r},\mu) = A_0(\mathbf{r})\frac{\psi_0(\mu)}{\mu} + \int_0^1 A_\nu\left(\mathbf{r}\right)\frac{\psi_\nu(\mu)}{\mu}d\nu. \tag{80}$$

The physical scalar flux is then obtained by integrating the pseudo-flux:

$$\Phi(\mathbf{r}) = \int_0^1 F(\mathbf{r},\mu)d\mu = A_0(\mathbf{r})\frac{1}{\sqrt{\nu_0 N_0}} + \int_0^1 A_\nu\left(\mathbf{r}\right)\frac{1}{\sqrt{\nu N_\nu}}d\nu, \tag{81}$$

(recall Eqs. (54)).

Remark. The unphysical character of the pseudo-flux $F(\mathbf{r},\mu)$ prevents, in general, from giving a direct meaning to a Dirichlet or Neumann condition at the boundary. Fortunately, it is often possible to extend a Cσ system to infinity by adding an extra region made of a purely absorbing material with the same total cross section as the other regions of the system, as observed in section 3. It can be shown that at the interface Σ between any two adjacent regions V_1, V_2 of a Cσ system, the familiar conditions of continuity

$$\begin{aligned} F_1(\mathbf{r}_\Sigma,\mu) &= F_2(\mathbf{r}_\Sigma,\mu), \\ \mu^2\frac{\partial F_1(\mathbf{r}_\Sigma,\mu)}{\partial n} &= \mu^2\frac{\partial F_2(\mathbf{r}_\Sigma,\mu)}{\partial n} \end{aligned} \tag{82}$$

rigorously hold for the pseudo-flux and its normal derivative (see [3]; the extension to the limit case $N \to \infty$ is immediate). This allows to introduce a coupling between all regions, much in the same way as for the multigroup diffusion equations. The simultaneous solution of as many boundary integral equations of the type of Eq. (72) as are the regions provides, in general, the boundary values of both F and $\partial F/\partial n$ to be used to evaluate $F(\mathbf{r},\mu)$ (and $\Phi(\mathbf{r})$) inside each region.

Let, for instance, V_1 be a finite, convex, homogeneous region with $\sigma_1 = 1$ and V_2 be the above extra-region (actually the complement of V_1 with respect to $\Re^3$). As V_2 is purely absorbing, the fundamental solution for it is given by Eq. (55), i.e.

$$\psi\left(\left|\mathbf{r}-\mathbf{r}'\right|,\mu,\mu'\right) = \frac{1}{\mu^2}\Psi_\mu(|\mathbf{r}-\mathbf{r}'|)\delta\left(\mu-\mu'\right), \tag{83}$$

using (56). The boundary integral equation for the sourceless region V_2 becomes, after performing the μ' integral,

$$\left[1-c\left(\mathbf{r}_\Sigma\right)\right]\xi(\mathbf{r}_\Sigma,\mu) + \int\limits_\Sigma \left[\Psi_\mu(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|)\frac{\partial\xi(\mathbf{r}'_\Sigma,\mu)}{\partial n'} - \frac{\partial\Psi_\mu(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|)}{\partial n'}\xi(\mathbf{r}'_\Sigma,\mu)\right]d\Sigma' = 0, \tag{84}$$

where it has been observed that the c factor of V_2 is the complement of the corresponding factor of V_1 with respect to unity and that $\mathbf{n}_2$, the outward normal of V_2 at the interface Σ with V_1, is equal to $-\mathbf{n}_1$, or simply $-\mathbf{n}$. Moreover, the limit values of F and $\partial F/\partial n$, as a point $\mathbf{r}_\Sigma$ is approached by interior points $\mathbf{r}$ of either region, turn out to be the same, owing to Eqs. (82). This is also true for ξ and $\partial\xi/\partial n$. Equations (72) and (84) then constitute a system of two integral equations yielding $\xi(\mathbf{r}_\Sigma,\mu)$ and $\partial\xi(\mathbf{r}_\Sigma,\mu)/\partial n$, the boundary quantities which are required in order to determine $\xi(\mathbf{r},\mu)$ (and $F(\mathbf{r},\mu)$) in V_1 and also in V_2, if useful.

The condition that V_1 is finite can be often suppressed, as in the following example.

Example. Let the region V_1 coincide with the half-space $x > 0$, where a source distribution $S(x)$ sustains the angular flux $\phi(x,\mu)$. It is readily seen that the kernel $\partial\psi(|\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma|,\mu,\mu')/\partial n'$ vanishes when $\mathbf{r}_\Sigma$ and $\mathbf{r}'_\Sigma$ belong to a plane boundary (or to a piece of the boundary that is a part of a plane), since the scalar product of the vectors $\mathbf{r}_\Sigma-\mathbf{r}'_\Sigma$ and $\mathbf{n}'$ is then equal to zero. Then the 1D boundary integral equation that corresponds to Eq. (37) is simply (note that $\partial/\partial n = -\partial/\partial x$ and that $c = 1/2$)

$$\frac{1}{2}F(0,\mu) + \int_0^1 d\mu' \psi_{pl}(0,\mu,\mu')\mu'^2 \frac{\partial F}{\partial x}(0,\mu) = Q(0,\mu), \tag{85}$$

while the equation corresponding to Eq. (84), written in terms of the pseudo-flux $F = \xi/\mu$, is

$$\frac{1}{2}F(0,\mu) - \psi^*_{pl}(0,\mu)\mu^2 \frac{\partial F}{\partial x}(0,\mu) = 0. \tag{86}$$

Now, $\psi^*_{pl}(0,\mu) = 1/(2\mu)$, by Eq. (48), so that the above equation can be written:

$$F(0,\mu) - \mu \frac{\partial F}{\partial x}(0,\mu) = 0, \tag{87}$$

and is therefore coincident with the classical void condition for the even-parity transport equation (this is, however, a fortunate circumstance which occurs only with 1D problems [5]).

Substitution into Eq. (85) leads to the following integral equation for $F(0,\mu)$:

$$\frac{1}{2}F(0,\mu) + \int_0^1 \psi_{pl}(0,\mu,\mu')F(0,\mu')\mu' d\mu' = Q(0,\mu). \tag{88}$$

We show that this equation is equivalent to the surface source integral equation of section 2, Eq. (19), with $\phi_\infty(0,\mu)$ given by Eq. (23).

We first observe that an equation similar to Eq. (19) can be written for the positive μ's. As the angular flux entering the boundary plane at $x = 0$ from the half-space $x < 0$ is zero (all the neutrons exiting from the half-space $x > 0$ are immediately suppressed by the negative surface sources), the equation is:

$$0 = \phi_\infty(0,\mu) - \int_{-1}^0 G(0,\mu' \to 0^+,\mu)\phi(0,\mu')\left|\mu'\right| d\mu', \qquad \mu \in (0,1]\,. \tag{89}$$

More generally, for any $x < 0$ one has

$$0 = \phi_\infty(x,\mu) - \int_{-1}^0 G(0,\mu' \to x,\mu)\phi(0,\mu')\left|\mu'\right| d\mu', \tag{90}$$

and, in particular, as $x \to 0^-$,

$$0 = \phi_\infty(0,\mu) - \int_{-1}^0 G(0,\mu' \to 0^-,\mu)\phi(0,\mu')\left|\mu'\right| d\mu', \tag{91}$$

no matter if μ is positive or negative. The following sequence of identities is easily checked:

$$
\begin{aligned}
G(0,\mu' \to 0^-,\mu) &= G(0^-,-\mu \to 0,-\mu') && \text{(reciprocity)},\\
&= G(0,-\mu \to 0^+,-\mu') && \text{(infinitesimal translation to right)},\\
&= G(0,-\mu' \to 0^+,-\mu) && \text{(symmetry with respect to } \mu,\ \mu'\text{, Eq. (9))},
\end{aligned}
\tag{92}
$$

so that Eq. (91) can be written as follows:

$$
0 = \phi_\infty(0,\mu) - \int_{-1}^{0} G(0,-\mu' \to 0^+,-\mu)\phi(0,\mu')\,|\mu'|\,d\mu'. \tag{93}
$$

A change of variables ($\mu \to -\mu$ and/or $\mu' \to -\mu'$) then allows to set Eqs. (19), (89) and (93), the latter one written separately for $\mu < 0$ and $\mu > 0$, in the following form:

$$
\begin{aligned}
\phi(0,-\mu) &= \phi_\infty(0,-\mu) - \int_0^1 G(0,-\mu' \to 0^+,-\mu)\phi(0,-\mu')\mu' d\mu',\\
0 &= \phi_\infty(0,\mu) - \int_0^1 G(0,-\mu' \to 0^+,\mu)\phi(0,-\mu')\mu' d\mu',\\
0 &= \phi_\infty(0,-\mu) - \int_0^1 G(0,\mu' \to 0^+,\mu)\phi(0,-\mu')\mu' d\mu',\\
0 &= \phi_\infty(0,\mu) - \int_0^1 G(0,\mu' \to 0^+,-\mu)\phi(0,-\mu')\mu' d\mu',
\end{aligned}
\tag{94}
$$

with $0 < \mu, \mu' \leq 1$. By summing up, using Eq. (42) for $\psi_{pl} = \phi^+$ and observing that $F(0,\mu) = \phi(0,\mu) + \phi(0,-\mu) = \phi(0,-\mu)$, $Q(0,\mu) = \phi_\infty(0,\mu) + \phi_\infty(0,-\mu)$, Eq. (88) is recovered, which concludes the proof.

Remark. Our present second-order theory deals only with isotropic volume sources. This explains why we have adopted the free term given by Eq. (23), instead of the one referring to the albedo problem.

5. CONCLUDING REMARKS

After recalling some general features of the A_N-SP_N method that are of importance for the A_∞-SP_∞ limit case, the A_∞ integrodifferential equation, an equivalent form of the transport equation which, although limited to the constant total cross section and isotropic scattering problems, has a simple and suggestive structure, is investigated, with emphasis on the relations with other topics of neutron transport. The A_∞ equation is transformed into a boundary integral form and the explicit structure of the fundamental solution (Green function) there involved is shown to determine a representation of the solution in terms of the Case eigenfunctions.

Some problems remain open. The first is represented by the artificial character of the pseudo-directional parameter μ and the corresponding pseudo angular flux $F(\mathbf{r},\mu)$. It can-not probably be avoided. Moreover, obtaining Case-like fully analytical solutions of realistic 2D and 3D problems seems to be hopeless, due to the involved structure of the boundary integral equations. What we have tried to do with this work is to explore a boundary region of ”Case’s kingdom”. By this respect, it appears that Case’s approach is more than a 1D (essentially plane geometry) theory. It is a 3D, A_∞-SP_∞ theory.

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