MULTI-POINT KINETICS EQUATIONS USING GENERALIZED PERTURBATION THEORY

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ABSTRACT

If we use an unperturbed flux to calculate kinetics parameters for multi-point kinetics equations which are derived using regionwise importance functions to produce fission neutrons, they contain an error of first order introduced by ignoring the change of the flux. Applying the generalized perturbation theory, a formulation to eliminate this error is derived. By numerical calculations for a simple system where an analytical solution can be easily obtained, it is shown that the accuracy of the derived equations is improved appreciably.

1. INTRODUCTION

It has been shown that by using regionwise importance functions representative producing fission neutrons, rigorous multi-point kinetics equations can be derived [1]∼[3]. Unknown functions of these kinetics equations have a clear physical meaning of the number of fission neutrons produced and the number of delayed neutron precursors in each region. The coupling coefficients k_{mn} and the kinetics parameters l_m used have a physical meaning of the number of fission neutrons produced in region V_m by a neutron born in region V_n and the regionwise neutron generation time, respectively [4],[5]. This feature is in contrast to the current one-point kinetics equation, where kinetics parameters are calculated with the weighting function of the adjoint flux, and the unknown function and kinetics parameters have no clear physical meanings[6].

If the derived multi-point kinetics equations are regarded as kinetics equations for coupled reactors, unknown functions and coupling coefficients have just the same physical meanings as those for the coupled reactors derived by Avery[7]. He seemed to have derived them by physical intuition approximately, however, using the regionwise importance functions, multi-point kinetics equations can be derived rigorously.

In order to calculate kinetics parameters of the multi-point kinetics equations for a perturbed system exactly, the exact flux for the perturbed system must be used. A disadvantage of these multi-point kinetics equations is the fact that a first order error is introduced in the kinetics parameters, if the flux for the unperturbed system is used, whereas the conventional one-point kinetics equations have an advantage that the first order error can be eliminated by using the adjoint flux as a weighting function.

In previous work [8],[9], the generalized perturbation theory(GPT)[10] was applied to obtain an asymptotic solution, and it has been shown that this first order error due to the use of the flux for unperturbed systems can be eliminated. In the present work, the GPT is used to obtain time dependent solutions of the multi-point kinetics equations; thus the present work is an extension of the previous work from the ω mode to the time domain for multi-point kinetics equations.

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2. MULTI-POINT KINETICS EQUATIONS

For simplicity, but without loss of generality, we consider the case where delayed neutrons are ignored. We assume that there is no external source, and the following multi-group diffusion equation holds for an unperturbed system:

$$A\psi_g(\boldsymbol{r}) = \frac{1}{k}B\psi_g(\boldsymbol{r}), \tag{1}$$

where k and $\psi_g(\boldsymbol{r})$ are a criticality factor and neutron flux of $g-$th group, respectively. The destruction and production operators A and B are defined by

$$A = -\nabla D_g \nabla + \Sigma_{rg} - \sum_{g' \neq g} \Sigma_s(g \leftarrow g'), \quad \text{and} \quad B = \chi_g F, \quad F = \sum_{g'} \nu\Sigma_{fg'}, \tag{2}$$

respectively, where D_g, χ_g, Σ_{rg}, $\Sigma_s(g \leftarrow g')$ and $\nu\Sigma_{fg'}$ are the diffusion coefficient, normalized fission spectrum, removal cross section, scattering cross section from group g' to group g and fission cross section multiplied by the total number of the fission neutrons, respectively.

We assume that neutron flux change due to a perturbation is expressed by the following time dependent multi-group diffusion equations;

$$\frac{1}{v_g}\frac{\partial\phi_g(\boldsymbol{r},t)}{\partial t} = H'\phi_g(\boldsymbol{r},t), \tag{3}$$

where prime denotes a quantity for the perturbed system, namely, H' means the perturbed one of the operator H;

$$H' = H + \delta H, \quad H = -A + \frac{1}{k}B, \quad \delta H' = -\delta A + \frac{1}{k}\delta B, \quad \delta B = \chi_g \delta F. \tag{4}$$

In order to derive N-point kinetics equations, we define regionwise importance functions $G_{mg}(\boldsymbol{r})$ by the equation,

$$A^\dagger G_{mg}(\boldsymbol{r}) = \nu\Sigma_{fg}\delta_m(\boldsymbol{r}), \quad m = 1, 2, .., N, \tag{5}$$

where $A^\dagger$ is an adjoint operator of operator A and

$$\delta_m(\boldsymbol{r}) = \begin{cases} 1, & \boldsymbol{r} \in V_m, \\ 0, & \boldsymbol{r} \notin V_m. \end{cases} \tag{6}$$

Multiplying Eq.(3) by this importance function and integrating it over the whole space domain and summing up over all energy groups, the following multi-point kinetics equations can be derived[3]:

$$l_m(t)\frac{ds_m(t)}{dt} = \sum_{n=1}^{N} \rho_{mn}(t)s_n(t), \tag{7}$$

where $s_m(t)$, $l_m(t)$ and $\rho_{mn}(t)$ are the regionwise integrated fission source, mean generation time and reactivity for each region V_m defined by

$$s_m(t) = \int_{V_m} s(\boldsymbol{r},t)d\boldsymbol{r}, \quad s(\boldsymbol{r},t) = F'\phi_g(\boldsymbol{r},t) = (F + \delta F)\phi_g(\boldsymbol{r},t), \tag{8}$$

$$l_m(t) = \frac{\int_V d\boldsymbol{r} \sum_g G_{mg}(\boldsymbol{r}) \frac{1}{v_g}\frac{\partial\phi_g(\boldsymbol{r},t)}{\partial t}}{\int_{V_m} d\boldsymbol{r} \frac{\partial F'\phi_g(\boldsymbol{r},t)}{\partial t}}, \tag{9}$$

$$\rho_{mn}(t) = \frac{1}{k}k_{mn}(t) + \Delta k_m^F(t)\delta_{mn} - \Delta k_{mn}^A(t) - \delta_{mn}, \tag{10}$$

respectively. The coupling coefficients and direct changes of coupling coefficients $\Delta k^A_{mn}(t)$ and $\Delta k^F_m(t)$ due to the perturbation are defined by

$$k_{mn}(t) = \frac{\int_{V_n} d\boldsymbol{r} \sum_g G_{mg}(\boldsymbol{r})\chi_g F'\phi_g(\boldsymbol{r},t)}{\int_{V_m} d\boldsymbol{r} F'\phi_g(\boldsymbol{r},t)}, \tag{11}$$

$$\Delta k^A_{mn}(t) = \frac{\int_{V_n} d\boldsymbol{r} \sum_g G_{mg}(\boldsymbol{r})\delta A\phi_g(\boldsymbol{r},t)}{\int_{V_n} d\boldsymbol{r} F'\phi_g(\boldsymbol{r},t)}, \quad \text{and} \quad \Delta k^F_m(t) = \frac{\int_{V_m} d\boldsymbol{r}\delta F\phi_g(\boldsymbol{r},t)}{\int_{V_m} d\boldsymbol{r} F'\phi_g(\boldsymbol{r},t)}, \tag{12}$$

respectively.

3. APPLICATION OF GENERALIZED PERTURBATION THEORY

We write the neutron flux of (3) in a form,

$$\phi_g(\boldsymbol{r},t) = \psi_g(\boldsymbol{r},t) + \delta\phi_g(\boldsymbol{r},t), \qquad \psi_g(\boldsymbol{r},t) = \psi_g(\boldsymbol{r})T(t), \tag{13}$$

where $\psi_g(\boldsymbol{r})$ is the solution of Eq.(1) and $\delta\phi_g(\boldsymbol{r},t)$ is an error from the exact flux of Eq.(3). We write the coupling coefficient $k_{mn}(t)$ in a form

$$k_{mn}(t) = \frac{\int_{V_n} d\boldsymbol{r} \sum_{g'} G_{mg'}(\boldsymbol{r})\chi_{g'} F'\phi_g(\boldsymbol{r},t)}{\int_{V_n} d\boldsymbol{r} F'\phi_g(\boldsymbol{r},t)} = \frac{\int_{V_n} d\boldsymbol{r} \sum_g \Sigma'_{\alpha mg}(\boldsymbol{r})\phi_g(\boldsymbol{r},t)}{\int_{V_n} d\boldsymbol{r} \sum_g \Sigma'_{\beta g}\phi_g(\boldsymbol{r},t)}, \tag{14}$$

where

$$\Sigma'_{\alpha mg}(\boldsymbol{r}) = \sum_{g'} G_{mg'}(\boldsymbol{r})\chi_{g'}\nu\Sigma'_{fg}(\boldsymbol{r}), \quad \text{and} \quad \Sigma'_{\beta g}(\boldsymbol{r}) = \nu\Sigma'_{fg}(\boldsymbol{r}). \tag{15}$$

Using these notations and retaining first order terms of the perturbation, we write Eq.(14) as

$$\begin{aligned} k_{mn}(t) &= \frac{\langle(\Sigma_{\alpha m} + \delta\Sigma_{\alpha m})(\psi + \delta\phi)\rangle_n}{\langle(\Sigma_\beta + \delta\Sigma_\beta)(\psi + \delta\phi)\rangle_n} = \frac{\langle\Sigma_{\alpha m}\psi + \Sigma_{\alpha m}\delta\phi + \delta\Sigma_{\alpha m}\psi + \delta\Sigma_{\alpha m}\delta\phi\rangle_n}{\langle\Sigma_\beta\psi + \Sigma_\beta\delta\phi + \delta\Sigma_\beta\psi + \delta\Sigma_\beta\delta\phi)\rangle_n} \\ &\doteqdot \frac{\langle\Sigma_{\alpha m}\psi\rangle_n}{\langle\Sigma_\beta\psi\rangle_n}\left(1 + \frac{\langle\Sigma_{\alpha m}\delta\phi\rangle_n}{\langle\Sigma_{\alpha m}\psi\rangle_n} + \frac{\langle\delta\Sigma_{\alpha m}\psi\rangle_n}{\langle\Sigma_{\alpha m}\psi\rangle_n} - \frac{\langle\Sigma_\beta\delta\phi\rangle_n}{\langle\Sigma_\beta\psi\rangle_n} - \frac{\langle\delta\Sigma_\beta\psi\rangle_n}{\langle\Sigma_\beta\psi\rangle_n}\right), \end{aligned} \tag{16}$$

where $\langle\Sigma_{\alpha m}\psi\rangle_n = \int_{V_n} d\boldsymbol{r} \sum_g \Sigma_{\alpha mg}(\boldsymbol{r})\psi_g(\boldsymbol{r})$. In order to calculate the first order terms of the change of the flux due to the perturbation, $\langle\Sigma_{\alpha m}\delta\phi\rangle_n$ and $\langle\Sigma_\beta\delta\phi\rangle_n$, we use the GPT.

We define the following quantity:

$$q_m(\boldsymbol{r}) = \left(\frac{\Sigma_{\alpha m}(\boldsymbol{r})}{\langle\Sigma_{\alpha m}\psi\rangle_n} - \frac{\Sigma_\beta(\boldsymbol{r})}{\langle\Sigma_\beta\psi\rangle_n}\right)\delta_n(\boldsymbol{r}). \tag{17}$$

From this definition, we can know that the following equation holds:

$$\int_{V_n} d\boldsymbol{r}\psi(\boldsymbol{r})q_m(\boldsymbol{r}) = \langle\psi(\boldsymbol{r})q_m(\boldsymbol{r})\rangle_n = 0. \tag{18}$$

Using Eq.(17), Eq.(16) can be written in a form,

$$k_{mn}(t) \doteqdot \frac{\langle\Sigma_{\alpha m}\psi\rangle_n}{\langle\Sigma_\beta\psi\rangle_n}\left(1 + \frac{\langle\delta\Sigma_{\alpha m}\psi\rangle_n}{\langle\Sigma_{\alpha m}\psi\rangle_n} - \frac{\langle\delta\Sigma_\beta\psi\rangle_n}{\langle\Sigma_\beta\psi\rangle_n} + \langle q_m\delta\phi\rangle_n\right). \tag{19}$$

Using an adjoint operator $H^\dagger$ of operator H and q_m of Eq.(17), we consider the following equation:

$$H^\dagger\phi^\dagger_{qm} = q_m. \tag{20}$$

Multiplying Eq.(20) by $\psi_g(\boldsymbol{r})$ of Eq.(1), integrating it over region V_n and using Green's formula, we obtain the following equation:

$$\left\langle \psi H^\dagger \phi^\dagger_{qm} \right\rangle_n = \left\langle \phi^\dagger_{qm} H \psi \right\rangle_n + B_{mn} = \langle \psi q_m \rangle_n = 0, \tag{21}$$

where

$$B_{mn} = \sum_g \int_{S_n} \left(\psi_g(\boldsymbol{r}) D_g \boldsymbol{n} \cdot \nabla \phi^\dagger_{gqm}(\boldsymbol{r}) - \phi^\dagger_{gqm}(\boldsymbol{r}) D_g \boldsymbol{n} \cdot \nabla \psi_g(\boldsymbol{r}) \right) dS. \tag{22}$$

Here, S_n and $\boldsymbol{n}$ are a surface and an outward unit vector normal to the surface of region V_n, respectively. The boundary condition for $\phi^\dagger_{qm}(\boldsymbol{r})$ is obtained from Eq.(21) such that the boundary term B_{mn} vanishes. Using Eq.(20), we obtain

$$\langle q_m \delta\phi \rangle_n = \left\langle \delta\phi H^\dagger \phi^\dagger_{qm} \right\rangle_n = \left\langle \phi^\dagger_{qm} H \delta\phi \right\rangle_n . \tag{23}$$

Substituting (13) into Eq.(3) and neglecting differential terms of $\delta\phi_g(\boldsymbol{r}, t)$ with respect to time and 2nd order terms of the perturbation which are made in deriving conventional one-point kinetics equations, we obtain

$$\frac{1}{v_g} \frac{\partial \psi_g(\boldsymbol{r}, t)}{\partial t} = H\psi_g(\boldsymbol{r}, t) + \delta H \psi_g(\boldsymbol{r}, t) + H \delta\phi_g(\boldsymbol{r}, t). \tag{24}$$

Multiplying Eq.(24) by $\phi^\dagger_{qm}$ and integrating it over region V_n, we obtain

$$\frac{\partial}{\partial t} \left\langle \frac{1}{v_g} \phi^\dagger_{qm} \psi_g(\boldsymbol{r}, t) \right\rangle_n = \left\langle \phi^\dagger_{qm} H \psi_g(\boldsymbol{r}, t) \right\rangle_n + \left\langle \phi^\dagger_{qm} \delta H \psi_g(\boldsymbol{r}, t) \right\rangle_n + \left\langle \phi^\dagger_{qm} H \delta\phi_g(\boldsymbol{r}, t) \right\rangle_n . \tag{25}$$

Using Eq.(1), the first term of the right hand side of the above equation vanishes. The differential term of the left hand side can vanish, if we use the condition

$$\left\langle \frac{1}{v_g} \phi^\dagger_{qm} \psi_g(\boldsymbol{r}, t) \right\rangle_n = 0, \tag{26}$$

when we obtain $\phi^\dagger_{qm}$ from Eq.(20). Then, from Eq.(25), the following equation results:

$$\left\langle \phi^\dagger_{qm} H \delta\phi_g(\boldsymbol{r}, t) \right\rangle_n = - \left\langle \phi^\dagger_{qm} \delta H \psi_g(\boldsymbol{r}, t) \right\rangle_n . \tag{27}$$

Using Eqs.(23) and Eq.(27), the coupling coefficients of Eq.(19) can be written as

$$k_{mn}(t) \doteqdot \frac{\langle \varSigma_{\alpha m} \psi \rangle_n}{\langle \varSigma_\beta \psi \rangle_n} \left(1 + \frac{\langle \delta\varSigma_{\alpha m} \psi \rangle_n}{\langle \varSigma_{\alpha m} \psi \rangle_n} - \frac{\langle \delta\varSigma_\beta \psi \rangle_n}{\langle \varSigma_\beta \psi \rangle_n} - \left\langle \phi^\dagger_{qm} \delta H \psi_g \right\rangle_n \right). \tag{28}$$

From Eq.(28), coupling coefficients can be obtained with the accuracy including the first order change of the flux due to the perturbation.

4. LOCAL PERTURBATION

We consider here a simple system, where analytic solutions can be obtained easily, namely, we assume that the system is homogeneous, a one dimensional slab geometry and one energy group. We assume also that a local perturbation which will cause a spatial change of the flux distribution, has a form of the Dirac's delta function, namely $\delta A = \varDelta\varSigma_a W \delta(x - \xi)$, $\delta B = \varDelta\nu\varSigma_f W \delta(x - \xi)$, where W is a fictitious thickness of the perturbed region. Then, Eq.(3) to be solved becomes

$$\frac{1}{v} \frac{\partial \phi(x, t)}{\partial t} = D \frac{\partial^2 \phi(x, t)}{\partial x^2} + \left[-\varSigma_a + \frac{1}{k} \nu\varSigma_f + \left(\frac{1}{k} \varDelta\nu\varSigma_f - \varDelta\varSigma_a \right) W \delta(x - \xi) \right] \phi(x, t). \tag{29}$$

We use the following boundary condition and initial condition,

$$\phi(x,t)|_{x=\pm a/2} = 0, \qquad \phi(x,0) = \psi(x) = \cos B_1 x, \qquad B_1 = \frac{\pi}{a}. \tag{30}$$

4.1 EXACT SOLUTION

The analytic solution of Eq.(29) can be obtained easily using a Fourier series, namely, we assume the following form

$$\phi(x,t) = \sum_{l=1}^{L} T_l(t) \sin \frac{l\pi}{a}\left(x + \frac{a}{2}\right). \tag{31}$$

Substituting Eq.(31) into Eq.(29), we obtain

$$l_l \frac{dT_l(t)}{dt} = \sum_{m=1}^{L} \rho_{lm} T_m(t), \tag{32}$$

where

$$k_l = \frac{k_\infty}{1 + L^2 B_l^2}, \qquad k_\infty = \frac{\nu \Sigma_f}{\Sigma_a}, \qquad l_l = \frac{l_\infty}{1 + L^2 B_l^2}, \qquad l_\infty = \frac{1}{\Sigma_a v}, \qquad B_l = \frac{l\pi}{a},$$

$$\rho_{lm} = \left(\frac{1}{k} k_l - 1\right)\delta_{lm} - \frac{2(\Delta\Sigma_a - \frac{1}{k}\Delta\nu\Sigma_f)W}{\Sigma_a a(1 + L^2 B_l^2)} \sin B_l\left(\xi + \frac{a}{2}\right) \sin B_m\left(\xi + \frac{a}{2}\right). \tag{33}$$

Substituting $T_l(t) = T_l e^{\omega t}$ into Eq.(32), we obtain an exact solution of Eq.(29) as

$$\phi(x,t) = \sum_{l=1}^{L}\sum_{l'=1}^{L} T_{ll'} \sin B_l\left(x + \frac{a}{2}\right) e^{\omega_{l'} t}, \tag{34}$$

where the eigenvalues ω_l and expansion coefficients T_{lm} are obtained from the eigenvalue equations

$$\sum_{l'} \frac{\rho_{ll'}}{l_l} T_{l'j} = \omega_j T_{lj}, \qquad \det\left|\frac{\rho_{lj}}{l_l} - \omega\delta_{lj}\right| = 0, \tag{35}$$

together with the conditions

$$\sum_{j=1}^{L} T_{1j} = 1, \qquad \sum_{j=1}^{L} T_{lj} = 0, \qquad \text{for} \quad l = 2, 3, ..., L, \tag{36}$$

which are derived from the initial condition of Eq.(30).
Using the flux of Eq.(34), the exact fission source and its time integral I_m for region V_m are calculated as

$$s_m(t) = \int_{V_m} \nu\Sigma_f \phi(x,t)dx, \qquad I_m = \int_0^\infty s_m(t)dt. \tag{37}$$

4.1.1 Conventional One-Point Kinetics Equation

In the case of one group, Eq.(1) is self adjoint, and its adjoint solution is equal to the flux $\psi(x)$. Using the unperturbed flux $\psi(x) = \cos Bx$ of Eq.(1), the kinetics parameters for the conventional one-point kinetics equations, Λ and reactivity ρ are obtained as[6]

$$\Lambda = \frac{\int_{-a/2}^{a/2} \psi^\dagger(x)\frac{1}{v}\psi(x)dx}{\int_{-a/2}^{a/2} \psi^\dagger(x)F'\psi(x)dx} = \frac{1}{v(\nu\Sigma_f + 2\Delta\nu\Sigma_f W \cos^2 B\xi/a)}, \tag{38}$$

$$\rho = \frac{\int_{-a/2}^{a/2} \psi^\dagger(x)\left(-\delta A + \frac{1}{k}\delta B\right)\psi(x)dx}{\int_{-a/2}^{a/2} \psi^\dagger(x)F'\psi(x)dx} = \frac{2\left(-\Delta\Sigma_a + \frac{1}{k}\Delta\nu\Sigma_f\right)W\cos^2 B\xi}{\nu\Sigma_f a + 2\Delta\nu\Sigma_f W\cos^2 B\xi}, \tag{39}$$

from which an inverse of the period, ω is obtained

$$\omega = \frac{\rho}{\Lambda} = \frac{2}{a}\left(-\Delta\Sigma_a + \frac{1}{k}\Delta\nu\Sigma_f\right) vW\cos^2 B\xi. \tag{40}$$

The fission source and its time integral are calculated by

$$s_1(t) = s_1(0)e^{\omega t}, \qquad I_1 = \int_0^\infty s_1(t) = -s_1(0)\frac{1}{\omega}. \tag{41}$$

4.2 ONE-POINT KINETICS EQUATIONS

In the case of one region $N = 1$, the importance function of Eq.(5) can be obtained as[9]

$$G_1(x) = k_\infty\left(1 - \frac{\cosh\kappa x}{\cosh\frac{\kappa a}{2}}\right), \quad \text{for} \quad -\frac{a}{2} \leqq x \leqq \frac{a}{2}, \quad \text{where} \quad k_\infty = \frac{\nu\Sigma_f}{\Sigma_a}, \quad \kappa = \sqrt{\frac{\Sigma_a}{D}}. \tag{42}$$

4.2.1 Without GPT

First, we consider the case where the GPT is not used, namely, terms including $\delta\phi$ are simply ignored. Using the unperturbed flux $\psi(x) = \cos B_1 x$ of Eq.(1), l_1 and k_{11} of Eqs.(9) and (11) become

$$l_1 = \frac{\int_{-a/2}^{a/2} G(x)\frac{1}{v}\psi(x)dx}{\int_{-a/2}^{a/2} F'\psi(x)dx} = \frac{l_{eff}}{1 + \dfrac{\Delta\nu\Sigma_f W B_1}{2\nu\Sigma_f}\cos B_1\xi}, \qquad l_{eff} = \frac{l_\infty}{1 + L^2B_1^2}, \tag{43}$$

$$k_{11} = \frac{\int_{-a/2}^{a/2} G(x)(\nu\Sigma_f + \Delta\nu\Sigma_f W\delta(x-\xi))\psi(x)dx}{\int_{-a/2}^{a/2} F'\psi(x)dx} = \frac{2\nu\Sigma_f k_{eff}/B_1 + \Delta\nu\Sigma_f G(\xi)W\cos B_1\xi}{2\nu\Sigma_f/B_1 + \Delta\nu\Sigma_f W\cos B_1\xi}. \tag{44}$$

The direct changes of the coupling coefficients of Eqs.(12) become

$$\Delta k_{11}^A = \frac{\int_{-a/2}^{a/2} G(x)\Delta\Sigma_a W\delta(x-\xi)\psi(x)dx}{\int_{-a/2}^{a/2} F'\psi(x)dx} = \frac{G(\xi)\Delta\Sigma_a W\cos B_1\xi}{2\nu\Sigma_f/B_1 + \Delta\nu\Sigma_f W\cos B_1\xi}, \tag{45}$$

$$\Delta k_1^F = \frac{\int_{-a/2}^{a/2} \Delta\nu\Sigma_f W\delta(x-\xi)\psi(x)dx}{\int_{-a/2}^{a/2} F'\psi(x)dx} = \frac{\Delta\nu\Sigma_f W\cos B_1\xi}{2\nu\Sigma_f/B_1 + \Delta\nu\Sigma_f W\cos B_1\xi}. \tag{46}$$

Using these kinetics parameters in Eq.(10), we obtain the reactivity and then ω as

$$\rho_{11} = \frac{\left(\frac{1}{k}\Delta\nu\Sigma_f - \Delta\Sigma_a\right)WB_1G(\xi)\cos B_1\xi}{2\nu\Sigma_f + \Delta\nu\Sigma_f WB_1\cos B_1\xi}, \tag{47}$$

$$\omega = \frac{\rho_{11}}{l_1} = \frac{1}{2k_{keff}}v\left(\frac{1}{k}\Delta\nu\Sigma_f - \Delta\Sigma_a\right)WG(\xi)\cos B_1\xi. \tag{48}$$

4.2.2 With GPT

We consider a one-point model using the GPT. Eq.(20) for slab geometry becomes

$$\left(D\frac{d^2}{dx^2} - \Sigma_a + \frac{1}{k}\nu\Sigma_f\right)\phi_q^\dagger(x) = q(x), \quad \text{for} \quad -\frac{a}{2} \leqq x \leqq \frac{a}{2}. \tag{49}$$

The fictitious cross sections of Eq.(15) to calculate k_{mn} and source term $q(x)$ are

$$\Sigma_\alpha = G(x)\nu\Sigma_f,\quad \delta\Sigma_\alpha = G(x)\Delta\nu\Sigma_f W\delta(x-\xi),\quad \Sigma_\beta = \nu\Sigma_f,\quad \delta\Sigma_\beta = \Delta\nu\Sigma_f W\delta(x-\xi), \tag{50}$$

$$q(x) = \frac{G_1(x)\nu\Sigma_f}{\langle G_1(x)\nu\Sigma_f\psi(x)\rangle} - \frac{\nu\Sigma_f}{\langle \nu\Sigma_f\psi(x)\rangle} = \frac{B_1}{2}(1+L^2B_1^2)\left(1-\frac{\cosh\kappa x}{\cosh\frac{\kappa a}{2}}\right) - \frac{B_1}{2}. \tag{51}$$

Solving Eq.(49), we obtain the solution as

$$\phi_q^\dagger(x) = \frac{B_1}{2\nu\Sigma_f}G_1(x) - \frac{2\cos B_1 x}{a\Sigma_a(1+L^2B_1^2)}. \tag{52}$$

Using Eq.(52), the coupling coefficient of Eq.(28) is obtained as

$$k_{11} = k_{11}\left[1 + \frac{\Delta\nu\Sigma_f W B_1 G_1(\xi)\psi(\xi)}{2\nu\Sigma_f k_{11}} - \frac{\Delta\nu\Sigma_f W B_1\psi(\xi)}{2\nu\Sigma_f} - \left(\frac{1}{k}\Delta\nu\Sigma_f - \Delta\Sigma_a\right)W\phi_q^\dagger(\xi)\psi(\xi)\right]. \tag{53}$$

Using Eqs.(45), Eq.(46) and (53), the reactivity of Eq.(10) and ω become

$$\rho_{11} = \frac{1}{k}k_{11} + \Delta k_1^F - \Delta k_{11}^A - 1 = \frac{2\left(-\Delta\Sigma_a + \frac{1}{k}\Delta\nu\Sigma_f\right)W\psi^2(\xi)}{a\Sigma_a(1+L^2B_1^2)}, \tag{54}$$

$$\omega = \frac{\rho_{11}}{l_1} = \frac{2}{a}\left(-\Delta\Sigma_a + \frac{1}{k}\Delta\nu\Sigma_f\right)vW\cos^2 B\xi. \tag{55}$$

Although the neutron generation time l_1 of Eq.(43) and reactivity ρ_{11} of Eq.(54) are different from the kinetics parameters Λ of Eq.(38) and reactivity ρ of Eq.(39) of the conventional one-point kinetics equation, the inverse of the period ω of Eq.(55) is the same as the one of Eq.(40). Therefore, the time dependence of both solutions is the same, and their accuracy can be regarded as the same.

4.3 TWO-POINT KINETICS EQUATIONS

We consider the two-point model, namely, $N = 2$. Let region V_1 be $-a/2 \leqq x \leqq 0$ and region V_2 be $0 \leqq x \leqq a/2$. Solving Eq.(5), the following importance function is obtained;

$$\begin{aligned} G_1(x) &= b^-\sinh\kappa\left(x+\frac{a}{2}\right) - k_\infty\cosh\kappa\left(x+\frac{a}{2}\right) + k_\infty, \qquad -\frac{a}{2} \leqq x \leqq 0, \\ &= b^+\sinh\kappa\left(\frac{a}{2}-x\right), \qquad 0 \leqq x \leqq \frac{a}{2}, \end{aligned} \tag{56}$$

where

$$b^- = \frac{k_\infty}{2}\frac{1}{\cosh\frac{\kappa a}{4}}\frac{1}{\cosh\frac{\kappa a}{2}}\sinh\frac{3\kappa a}{4}, \qquad b^+ = \frac{k_\infty}{\sinh\kappa a}\left(\cosh\frac{\kappa a}{2} - 1\right). \tag{57}$$

We assume that a perturbation is introduced in region V_2, namely, $0 < \xi < a/2$.

4.3.1 Without GPT

First, we consider the model where the GPT is not used. Neutron generation time of Eq.(9) and coupling coefficients of Eq.(16) neglecting the terms of $\delta\phi$ are obtained as follows:

$$l_1 = \frac{\sum_{n=1}^{2}\left\langle G_1\frac{1}{v_1}\psi\right\rangle_n}{\left\langle \nu\Sigma'_f\psi\right\rangle_1} = \frac{l_\infty}{1+L^2B_1^2}, \tag{58}$$

$$k_{11} = \frac{\langle \Sigma_{\alpha 1}\psi\rangle_1}{\langle \Sigma_\beta \psi\rangle_1}\left(1+\frac{\langle \delta\Sigma_{\alpha 1}\psi\rangle_1}{\langle \Sigma_{\alpha 1}\psi\rangle_1}-\frac{\langle \delta\Sigma_\beta\psi\rangle_1}{\langle \Sigma_\beta\psi\rangle_1}\right) = \frac{\langle G_1(x)\nu\Sigma_f\cos B_1x\rangle_1}{\langle \nu\Sigma_f\cos B_1 x\rangle_1}$$
$$= \frac{k_\infty(2-LB_1\tanh\frac{\kappa a}{4})}{2(1+L^2B_1^2)}, \tag{59}$$
$$k_{12} = \frac{\langle \Sigma_{\alpha 1}\psi\rangle_2}{\langle \Sigma_\beta \psi\rangle_2}\left(1+\frac{\langle \delta\Sigma_{\alpha 1}\psi\rangle_2}{\langle \Sigma_{\alpha 1}\psi\rangle_2}-\frac{\langle \delta\Sigma_\beta\psi\rangle_2}{\langle \Sigma_\beta\psi\rangle_2}\right)$$
$$= \frac{k_\infty LB_1\tanh\frac{\kappa a}{4}}{2(1+L^2B_1^2)}\left(1+\frac{\Delta\nu\Sigma_f W G_1(\xi)\cos B_1\xi}{\nu\Sigma_f\langle G_1(x)\cos B_1x\rangle_2}-\frac{B_1\Delta\nu\Sigma_f W\cos B_1\xi}{\nu\Sigma_f}\right), \tag{60}$$
$$k_{21} = \frac{\langle \Sigma_{\alpha 2}\psi\rangle_1}{\langle \Sigma_\beta \psi\rangle_1}\left(1+\frac{\langle \delta\Sigma_{\alpha 2}\psi\rangle_1}{\langle \Sigma_{\alpha 2}\psi\rangle_1}-\frac{\langle \delta\Sigma_\beta\psi\rangle_1}{\langle \Sigma_\beta\psi\rangle_1}\right) = \frac{\langle G_2(x)\nu\Sigma_f\cos B_1x\rangle_1}{\langle \nu\Sigma_f\cos B_1 x\rangle_1}$$
$$= B_1\langle G_1(x)\cos B_1x\rangle_2 = \frac{k_\infty LB_1\tanh\frac{\kappa a}{4}}{2(1+L^2B_1^2)}, \tag{61}$$
$$k_{22} = \frac{\langle \Sigma_{\alpha 2}\psi\rangle_2}{\langle \Sigma_\beta \psi\rangle_2}\left(1+\frac{\langle \delta\Sigma_{\alpha 2}\psi\rangle_2}{\langle \Sigma_{\alpha 2}\psi\rangle_2}-\frac{\langle \delta\Sigma_\beta\psi\rangle_2}{\langle \Sigma_\beta\psi\rangle_2}\right)$$
$$= B_1\langle G_1(x)\cos B_1x\rangle_1\left(1+\frac{\Delta\nu\Sigma_f W G_1(-\xi)\cos B_1\xi}{\nu\Sigma_f\langle G_1(x)\cos B_1x\rangle_1}-\frac{B_1\Delta\nu\Sigma_f W\cos B_1\xi}{\nu\Sigma_f}\right), \tag{62}$$

where the following integrals are used:

$$\langle G_1(x)\cos B_1x\rangle_1 = \frac{k_\infty(2-LB_1\tanh\frac{\kappa a}{4})}{2B_1(1+L^2B_1^2)}, \qquad L=\frac{1}{\kappa}, \tag{63}$$
$$\langle G_1(x)\cos B_1x\rangle_2 = \frac{L\cosh\frac{\kappa a}{2}}{1+L^2B_1^2}b^+ = \frac{k_\infty L\tanh\frac{\kappa a}{4}}{2(1+L^2B_1^2)}. \tag{64}$$

Using the above results, Eqs.(10) and (12), we obtain reactivities

$$\rho_{11} = \frac{1}{k}k_{11}-1, \qquad \rho_{12} = \frac{1}{k}k_{12}-\Delta k_{12}^A, \tag{65}$$
$$\rho_{21} = \frac{1}{k}k_{21}-\Delta k_{21}^A, \qquad \rho_{22} = \frac{1}{k}k_{22}+\Delta k_2^F-\Delta k_{22}^A-1, \tag{66}$$
$$\Delta k_{12}^A = \frac{B_1\Delta\Sigma_a W G_1(\xi)\cos B_1\xi}{\nu\Sigma_f}, \qquad \Delta k_2^F = \frac{B_1\Delta\nu\Sigma_f W\cos B_1\xi}{\nu\Sigma_f},$$
$$\Delta k_{21}^A = \frac{B_1\Delta\Sigma_a W B_1 G_1(-\xi)\cos B_1\xi}{\nu\Sigma_f}, \qquad \Delta k_{22}^A = \frac{B_1\Delta\Sigma_a W G_1(-\xi)\cos B_1\xi}{\nu\Sigma_f}. \tag{67}$$

Making use of these kinetics parameters in Eq.(7), fission sources $s_m(t)$ are obtained as in the solution of Eq.(32) as

$$s_m(t) = \sum_{n=1}^{2} s_{mn}e^{\omega_n t}, \qquad \text{for} \quad m=1,2, \tag{68}$$

where eigenvalues ω_1, ω_2 and coefficients s_{mn} are obtained from the following eigenvalue equation:

$$\sum_{n'}\frac{\rho_{mn'}}{l_m}s_{n'n} = \omega_n s_{mn}, \qquad \det\left|\frac{\rho_{mn}}{l_m}-\omega\delta_{mn}\right| = 0. \tag{69}$$

4.3.2 With GPT

We consider two-point kinetics equations with the GPT. Since the unperturbed system is symmetric with respect to the origin, the importance function has a relation $G_2(x) = G_1(-x)$.

Therefore, we need to obtain only $G_1(x)$. The boundary term of Eq.(22) becomes

$$B_{mn} = \psi(x)D\frac{d\phi^\dagger_{qm}(x)}{dx}\Bigg|_{x=x_{n0}}^{x_{n1}} - \phi^\dagger_{qm}(x)D\frac{d\psi(x)}{dx}\Bigg|_{x=x_{n0}}^{x_{n1}}, \tag{70}$$

from which the boundary condition for $\phi^\dagger_{q1}(x)$ of the present problem is derived as

$$\phi^\dagger_{q1}(x)|_{x=-a/2} = 0, \qquad \frac{d\phi^\dagger_{q1}(x)}{dx}|_{x=0} = 0. \tag{71}$$

Applying this boundary condition and the condition of Eq.(26), solution of Eq.(20) is obtained:

$$\begin{aligned}\phi^\dagger_{q1}(\boldsymbol{r}) &= b_1^- \cos B_1 x + b_2^- \sin B_1 x + \phi_p^-(x), \qquad -\frac{a}{2} \leqq x < 0,\\ &= b_1^+ \cos B_1 x + b_2^+ \sin B_1 x + \phi_p^+(x), \qquad 0 < x \leqq \frac{a}{2},\end{aligned} \tag{72}$$

where $\phi_p^\pm(x)$ are the particular solutions of Eq.(20) and

$$\phi_p^\pm(x) = c_1^\pm G_1(x) + c_2^\pm, \tag{73}$$

$$b_1^- = \frac{4}{a}\left(\frac{b_2^-}{2B_1} - \left\langle\phi_p^-(x)\cos B_1 x\right\rangle_1\right), \qquad b_2^- = c_2^-, \tag{74}$$

$$c_1^- = \frac{1}{\Sigma_a(1+L^2B_1^2)\left\langle G_1(x)\cos B_1 x\right\rangle_1} = \frac{2B_1}{k_\infty\Sigma_a(2 - LB_1\tanh\frac{\kappa a}{4})}, \tag{75}$$

$$c_2^- = \frac{1}{DB_1^2}\left(\frac{k_\infty}{(1+L^2B_1^2)\left\langle G_1(x)\cos B_1 x\right\rangle_1} - B_1\right) = \frac{L\tanh\frac{\kappa a}{4}}{D(2 - LB_1\tanh\frac{\kappa a}{4})}, \tag{76}$$

$$b_1^+ = -\frac{4}{a}\left(\frac{b_2^+}{2B_1} + \left\langle\phi_p^+(x)\cos B_1 x\right\rangle_2\right), \qquad b_2^+ = -c_2^+, \tag{77}$$

$$c_1^+ = \frac{1}{\Sigma_a(1+L^2B_1^2)\left\langle G_1(x)\cos B_1 x\right\rangle_2} = \frac{2L}{Dk_\infty\tanh\frac{\kappa a}{4}}, \qquad c_2^+ = -\frac{1}{DB_1}. \tag{78}$$

Using this adjoint function, the coupling coefficients of Eq.(28) are obtained as

$$\begin{aligned}k_{12} &= \frac{\langle\Sigma_{\alpha1}\psi\rangle_2}{\langle\Sigma_\beta\psi\rangle_2}\left(1 + \frac{\langle\delta\Sigma_{\alpha1}\psi\rangle_2}{\langle\Sigma_{\alpha1}\psi\rangle_2} - \frac{\langle\delta\Sigma_\beta\psi\rangle_2}{\langle\Sigma_\beta\psi\rangle_2}\right)\\ &= \frac{k_\infty LB_1\tanh\frac{\kappa a}{4}}{2(1+L^2B_1^2)}\left(1 + \frac{\Delta\nu\Sigma_f W G_1(\xi)\cos B_1\xi}{\nu\Sigma_f\left\langle G_1(x)\cos B_1 x\right\rangle_2} - \frac{B_1\Delta\nu\Sigma_f W\cos B_1\xi}{\nu\Sigma_f}\right.\\ &\qquad \left. + (\Delta\Sigma_a - \frac{1}{k}\Delta\nu\Sigma_f)W\phi^\dagger_{q1}(\xi)\cos B_1\xi\right),\end{aligned} \tag{79}$$

$$\begin{aligned}k_{22} &= \frac{\langle\Sigma_{\alpha2}\psi\rangle_2}{\langle\Sigma_\beta\psi\rangle_2}\left(1 + \frac{\langle\delta\Sigma_{\alpha2}\psi\rangle_2}{\langle\Sigma_{\alpha2}\psi\rangle_2} - \frac{\langle\delta\Sigma_\beta\psi\rangle_2}{\langle\Sigma_\beta\psi\rangle_2}\right)\\ &= B_1\left\langle G_1(x)\cos B_1 x\right\rangle_1\left(1 + \frac{\Delta\nu\Sigma_f W G_1(-\xi)\cos B_1\xi}{\nu\Sigma_f\left\langle G_1(x)\cos B_1 x\right\rangle_1} - \frac{B_1\Delta\nu\Sigma_f W\cos B_1\xi}{\nu\Sigma_f}\right.\\ &\qquad \left. + (\Delta\Sigma_a - \frac{1}{k}\Delta\nu\Sigma_f)W\phi^\dagger_{q1}(-\xi)\cos B_1\xi\right).\end{aligned} \tag{80}$$

Other coupling coefficients are the same as those given in the previous subsection. Using these coupling coefficients in Eq.(7), we can obtain a solution which has the same form as Eq.(68).

5. NUMERICAL EXAMPLE

In order to investigate the accuracy of the equations derived in the preceding chapters, numerical calculations are performed. Constants $D = 1\text{cm}^{-1}$, $\Sigma_a = 0.1\text{cm}^{-1}$, $\nu\Sigma_f = 0.100987\text{cm}^{-1}$, $v = 2.2\times10^5\text{cm sec}^{-1}$, $a = 200\text{cm}$ are used. As a perturbation, we use $W = 1\text{cm}$, $\Delta\nu\Sigma_f = 0$ and $\Delta\Sigma_a = 0.1\Sigma_a$ at $\xi = 70\text{cm}$. Integrated fission sources are normalized such that $s_1(0) = s_2(0) = 1$ for both one-point and two-point models. In Figs.1 $\sim$ 4 are shown the importance functions of Eqs.(42) and (56), and the adjoint functions of Eqs.(52) and (72).

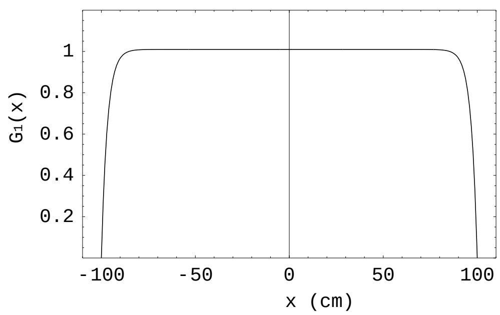

Figure 1. Importance Function $G_1(x)$ of Eq.(42) for one-point model.

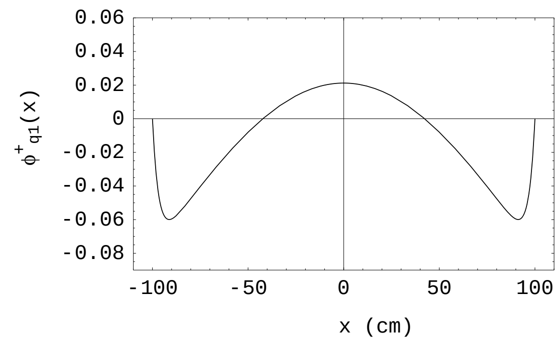

Figure 2. Adjoint function $\phi^\dagger_{q1}(x)$ of Eq.(52) for one-point model.

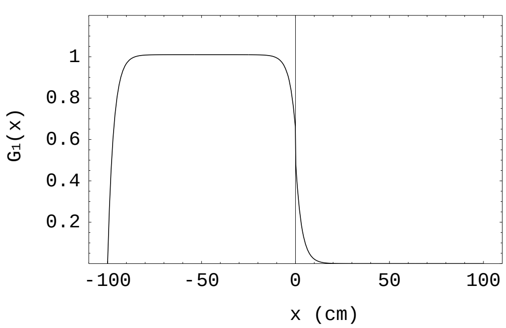

Figure 3. Importance Function $G_1(x)(= G_2(-x))$ of Eq.(56) for two-point model.

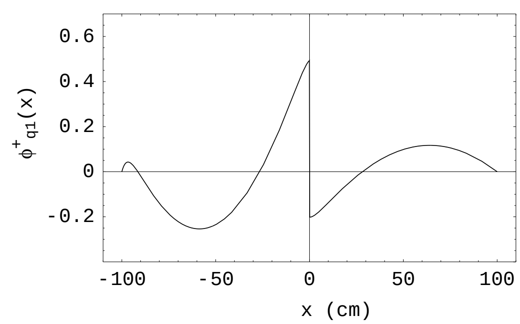

Figure 4. Adjoint function $\phi^\dagger_{q1}(x)$ of Eq.(72). For $0 < x \leqq a/2$, $0.01 \times \phi^\dagger_{q1}(x)$ is shown.

In Table I are shown reactivities calculated by Eqs.(47) and (54), ω by Eqs.(48) and (55) and integrated fission sources. The exact solution is calculated using a number of Fourier series of $L = 30$. From these values, we can know that the accuracy can be improved appreciably by using the GPT.

Table I. Reactivities and integrated fission sources for one-point model

	ρ_{11}	ω_1 (sec^{-1})	I_1
Without GPT	-0.000356536	-7.8632	16.19
With GPT	-0.000205600	-4.5344	28.08
Exact			33.3118

In Tables II and III are shown reactivities calculated using Eqs.(65) and (66) for the two-point model, ω_1 and ω_2 using Eq.(69) and integrated fission sources without and with the GPT. In this case also, the accuracy is improved very much by using GPT. In Figs.5 and 6 are shown fission sources for the two-point model with the GPT and the exact solution. It is seen that fission source $s_2(t)$ in region V_2 is less than $s_1(t)$ in region V_1, which is reasonable, since the perturbation of increasing absorption cross section is introduced in region V_2. Although the two point model can express more the realistic flux distribution than the one-point model, the accuracy of the two-point model is improved not so much compared with the one-point model.

Table II. Reactivities and ω_n for Two-point model

$\Delta\Sigma_a/\Sigma_a$	ρ_{11}	ρ_{12}	ρ_{21}	ρ_{22}	ω_1 (sec^{-1})	ω_2 (sec^{-1})
0	-0.024836	0.024836	0.024836	-0.024836	0	0
			Without GPT			
0.1	-0.024836	0.026119	0.024836	-0.025549	-7.807	-1103
			With GPT			
0.1	-0.024836	0.026119	0.024836	-0.026529	-4.402	-1128

Table III. Integrated fission sources for two-point model

	I_1	I_2	$I_1 + I_2$
Without GPT	8.212	8.096	16.31
With GPT	14.88	14.04	28.92
Exact	17.36	15.96	33.32

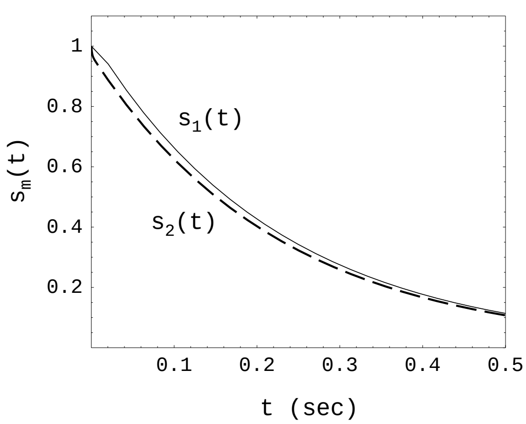

Figure 5. $s_1(t)$ and $s_2(t)$ with GPT for two-point model

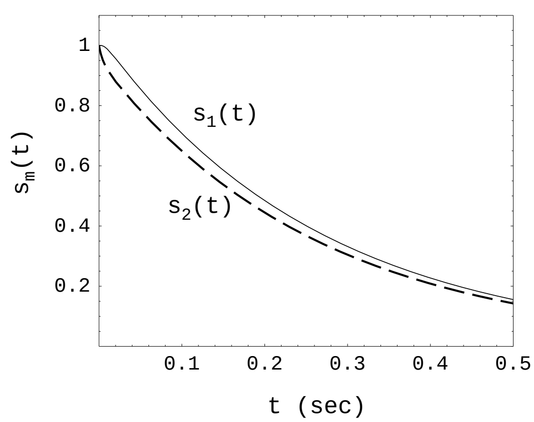

Figure 6. $s_1(t)$ and $s_2(t)$ of Eq.(37) of exact solution

CONCLUSION

Applying the GPT, kinetics parameters are derived in which the first order change of the flux is taken into account. Numerical calculations show that the accuracy of the kinetics equations using GPT is improved appreciably. However, the accuracy of the two-point model is not so much improved compared with the one-point model. Therefore, there remains a problem to derive equations for the two-point model or for multi-point models which are more accurate than the one-point model. Since the unknown functions and kinetics parameters of the present kinetics equations have clear physical meanings, a fictitious system expressed by the effective multiplication factor for subcritical systems needs not be considered. Therefore, the present equations can be used to explain the experimental results for subcritical systems with much reality.

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